Bayesian variable selection with a focus on the analysis of genomic data - Part II

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Bayes 2013
Challenges in High-dimensional Data

“High-dimensionality”: growing data dimension along with the sample size, where \( p_n > n \)

Important attributes of statistical procedures:
- accuracy of inference
- computational tractability

Challenges in high-dimensional data
- What is the sensible threshold of dimensionality to apply a statistical procedure?
- Characterization of optimality attributes
- Development of reliable inferential tools
- “Low assumptions in high dimensions”
High Dimensions $\sim$ Strong Assumptions

$\implies$ What is the convenient dimensionality of the parametrization?

$\implies$ A crucial assumption is the one of sparsity

- parametrization using only a few coefficients
- often in line with biological intuition

$\implies$ Sparsity central to the implementation of variable selection!

$\implies$ Bayesian variable selection: natural incorporation of the prior knowledge on the pattern of sparsity
Computational Aspects

⇝ Penalized likelihood methods: (LASSO, SCAD...)  
  ● easy for convex penalties  
  ● non-convex penalties  
  (Fan and Li (2009), Hunter and Li (2005))

⇝ Bayesian shrinkage methods: (Bayesian LASSO...)  
  ● MCMC with block updates  
  ● MAP estimation using EM algorithm  
  (Griffin and Brown (2005), Rockova and Lesaffre (2013))

⇝ Bayesian variable selection: (spike and slab, SSVS)  
  ● MCMC  
  (George and McCulloch (1993); Hans et al. (2010))  
  ● EM algorithm for posterior model mode detection  
  (Rockova and George (2012))
SSVS Setup

Assume $\mathbf{Y} \sim N_n(\alpha + \mathbf{X}\beta, \sigma^2\mathbf{I}_n)$, interest in $p > n$

Binary variable selection indicators $\gamma = (\gamma_1, \ldots, \gamma_p)'$, where
$\gamma_i = 0$ if $\beta_i$ is “small” and $\gamma_i = 1$ if $\beta_i$ is “large”

Conjugate “spike and slab” prior on regression coefficients

$$
\pi(\beta_i | \sigma, \gamma) = N(0, \sigma^2[(1 - \gamma_i)\nu_0 + \gamma_i\nu_1]),
$$

$\gamma_i = 0$: Spike variance $\sigma^2\nu_0$ small
$\gamma_i = 1$: Slab variance $\sigma^2\nu_1$ large

Prior distribution for the variance $\pi(\sigma^2 | \gamma) = IG(\nu/2, \nu\lambda/2)$

Uniform improper prior on the intercept $\alpha \sim margined out$
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Conjugate “spike and slab” prior on regression coefficients

\[ \pi(\beta_i | \sigma, \gamma) = N(0, \sigma^2[(1 - \gamma_i)v_0 + \gamma_i v_1]), \]

\[ \gamma_i = 0: \text{ Spike variance } \sigma^2 v_0 \text{ small} \]

\[ \gamma_i = 1: \text{ Slab variance } \sigma^2 v_1 \text{ large} \]

\[ \pi(\sigma^2 | \gamma) = IG(\nu/2, \nu\lambda/2) \]

Uniform improper prior on the intercept \[ \alpha \sim \text{ margined out} \]
SSVS Setup

Spike and slab prior for $\nu_0 = 0.01$, $\nu_1 = 0.1$
Stochastic Model Search

Favored selection criteria based on $\pi(\gamma | Y)$

(a) Highest posterior probability model

$$\arg\max_{\gamma} \pi(\gamma | Y)$$

(b) Median probability model: select variables with

$$P(\gamma_i = 1 | Y) > 0.5$$

MCMC stochastic search algorithms attempt to find these

- SSVS (George and McCulloch (1993))
- ESS (Botello and Richardson (2010))
- SSS (Hans et al. (2010))

Slow and inefficient, especially when $p$ is large.

Is there a better way?
Deterministic Model Search

Rockova and George (2012) propose an EM model search algorithm

(1) “\( \pi(\gamma \mid Y) \leftrightarrow \pi(\beta \mid Y) \)”

\[ \Rightarrow \text{High posterior modes of } \pi(\gamma \mid Y) \text{ can be located by thresholding small coefficient estimates of associated high posterior modes of } \pi(\beta \mid Y) \]

\[ \Rightarrow \text{Modes of the posterior } \pi(\beta \mid Y) \text{ can be found deterministically} \]

(2) “Spike-and-slab Regularization Diagram”

\[ \Rightarrow \text{Obtain modal estimates for a sequence of mixture priors with increasing } \nu_0 > 0 \]

\[ \Rightarrow \text{Depicts evolution and gradual sparsification of selected subsets} \]
Deterministic Model Search

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\(\leadsto\) High posterior modes of \(\pi(\gamma \mid Y)\) can be located by thresholding small coefficient estimates of associated high posterior modes of \(\pi(\beta \mid Y)\)

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(2) “Spike-and-slab Regularization Diagram”

\(\leadsto\) Obtain modal estimates for a sequence of mixture priors with increasing \(v_0 > 0\)

\(\leadsto\) Depicts evolution and gradual sparsification of selected subsets
SSVS $\rightarrow$ EMVS
Key Ingredients

(1) **Conjugacy**
   - Allows analytical simplifications for the EM algorithm
   - Enables computation of posterior model probabilities

(2) **Use of both** $v_0 > 0$ **and** $v_0 = 0$
   - $v_0 > 0$: Feasible closed form EM algorithm
   - $v_0 > 0$: Spike distribution absorbs small coefficients
   - $v_0 = 0$: Correct posterior for candidate model evaluation

(3) **$\pi(\gamma|\theta)$ Flexibility**
   - Allows incorporation of covariate pattern information
EMVS Algorithm

**GOAL** To locate posterior mode

\[
\text{argmax}_{\beta, \theta, \sigma} \log \pi(\beta, \theta, \sigma^2 | \mathbf{y})
\]  \hspace{1cm} (1)

**IDEA** Solve this via EM by treating \( \gamma \) as "missing data" and focusing on

\[
\log \pi(\beta, \theta, \sigma^2, \gamma | \mathbf{y})
\]

**E-step**

Compute conditional expectation of "log complete data posterior":

\[
Q \left( \beta, \theta, \sigma | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)} \right) = \mathbb{E}_{\gamma | \cdot} \left[ \log \pi(\beta, \theta, \sigma, \gamma | \mathbf{y}) | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, \mathbf{y} \right]
\]

**M-step**

Maximize \( Q \left( \beta, \theta, \sigma | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)} \right) \) to get \( (\beta^{(k+1)}, \theta^{(k+1)}, \sigma^{(k+1)}) \)
Simple Implementation

\[ \sim \quad \text{Assume } n = 100, \ p = 1000, \ \beta = (1, 2, 3, 0, \ldots, 0)' \]

\[ \sim \quad \text{Rows in } X \text{ sampled from } N_p(0, \Sigma), \text{ where } \Sigma = (0.6|i-j|)^{p}_{i,j=1} \]

\[ \sim \quad Y = X\beta + \varepsilon, \text{ where } \varepsilon \sim N_n(0, \sigma^2 I_p) \text{ and } \sigma^2 = 3 \]

\[ \sim \quad \text{For now fixed } \nu_0 = 1 \text{ and } \nu_1 = 1000, \ \theta \sim \text{Beta}(1, 1) \]
Simple Implementation

⇝ Subset selection for fixed $v_0$:

(a) **Based on conditional inclusion probabilities**

Select $X_i$ when $P(\gamma_i = 1 | \hat{\beta}, \hat{\sigma}, \hat{\theta}) > 0.5$

(b) **Based on modal estimates** $\hat{\beta}$ (equivalent to (a))

Select $X_i$ when $|\hat{\beta}_i| > \mu_{v_0, v_1, \hat{\sigma}} = \hat{\sigma} \sqrt{2v_0 \log(\omega_i c) c^2 / (c^2 - 1)}$

with $c^2 = v_1 / v_0$ and $\omega_i = [1 - P(\gamma_i = 1 | \hat{\theta})] / P(\gamma_i = 1 | \hat{\theta})$

⇝ We can consider a grid $V$ of values $v_0$ and $\forall v_0 \in V$ determine an active set $S_{v_0} = \{1 \leq i \leq p : |\hat{\beta}_i| > \mu_{v_0, v_1, \hat{\sigma}}\}$

⇝ For each active set we evaluate $\gamma$ using

$$g_{v_0}(\gamma) = C_{p}(\gamma | Y)$$

assuming that $v_0 = 0$ (correct submodel evaluation)
Simple Implementation Continued

\[ \text{Regularization plot for a grid of values } v_0 \]
\[ \text{Starting values } \beta^{(0)} = 1_p, \sigma^{(0)} = 1 \text{ and } \theta^{(0)} = 0.5 \]
\[ \text{Increasing } v_0 \text{ absorbs smaller coefficients} \]
Multimodality Issues

\[ \Rightarrow \text{EM algorithm guarantees monotonical convergence towards at least a local maximum} \]

\[ \Rightarrow \text{Prone to entrapment around local modes} \]

\[ \Rightarrow \text{Posterior from conjugate model with two correlated predictors, } \beta = (1, 0)', v_1 = 1000 \text{ and } v_0 = 0.005, \hat{\beta}_{MLE} = (0.52, 0.4)' \]
Deterministic Annealing

\( \Rightarrow \) Maximize a tempered version of the objective function: for \( 0 < t < 1 \)

\[
H_t(\beta, \theta, \sigma) = \frac{1}{t} \log \sum_{\gamma} \pi(\beta, \theta, \sigma, \gamma | y)^t
\]  \hspace{1cm} (2)

\( \Rightarrow \) Temperature \( 1/t \) regulates the degree of separation between multiple modes

\( \Rightarrow \) Small values \( t \) smooth the function to have only one mode

\( \Rightarrow \) Consider temperature ladder \( 1/t_1 < 1/t_2 < \cdots < 1/t_T \)

\( \Rightarrow \) Solutions at lower temperature can be used as starting points for computation at higher temperature
**Simple Implementation Continued**

~~ Regularization plot for a grid of values $\nu_0, \nu_1 = 1000$~~

~~ Randomly generated starting values $\beta^{(0)} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I})$~~

Temperature 1

Temperature 10

EMVS Regularization Plot
Structured Model Priors

Variable selection indicators assigned prior distribution $\pi(\gamma | \theta)$

(a) **Beta-binomial prior** (George and McCulloch (1993))

$$\pi(\gamma | \theta) = \theta \sum \gamma_i (1 - \theta)^{p - \sum \gamma_i} \quad \text{with} \quad \theta \sim B(a, b)$$

(b) **Logistic regression product prior** (Stingo et al. (2010))

$$\pi(\gamma | \theta) = \prod_{i=1}^{p} \left( \frac{\exp(Z_i \theta)}{1 + \exp(Z_i \theta)} \right)^{\gamma_i} \left( \frac{1}{1 + \exp(Z_i \theta)} \right)^{1-\gamma_i}$$

(c) **Markov random field prior** (Li and Zhang (2010))

$$\pi(\gamma | \theta) = \exp \left[ \theta'_1 \gamma + \gamma' \theta_2 \gamma - \psi(\theta_1, \theta_2) \right]$$
Simulated Example with Structured Covariates

_ASSUME p = 99 covariates cluster within three non-overlapping groups: Z = [z₁, z₂, z₃], where zᵢⱼ = I[33(i−1)+1; 33i](j)

_ASSUME within group correlation 0.8, between group correlation 0

_ASSUME Responses Y ∼ N(Xβ, σ²Iₙ) with β = 2 × I[1;33](i), n = 100 and σ² = 5

_ASSUME EMVS with (a) Beta-binomial prior, (b) logistic regression prior, (c) MRF prior

Settings for model exploration

- ν₀ ∈ {0.01 + k × 0.5 : 0 ≤ k ≤ 10}
- ν₁ assigned prior (6) with a = 0.5 and b = 250

Settings for model evaluation

- ν₁ fixed to 1 000
- Uniform beta-binomial for the g-function
Simulated Example with Structured Covariates

(a) Beta-binomial prior, where $\theta \sim B(1, 1)$

Best visited model:

25 true positives together with 11 false negatives and 8 false positives
Simulated Example with Structured Covariates

(b) Logistic regression prior $\theta \sim \pi(\theta)$ in (3) with $a = b = 1$

Best visited model:

28 true positives together with 5 false negatives and 0 false positives

EMVS Regularization Plot

Log($g(\gamma)$)
Simulated Example with Structured Covariates

(c) MRF prior, $\theta$ fixed to the phase transition point, $\theta_2 = (1_{33 \times 33} - I_{33}) \otimes I_3$

Best visited model:
33 true positives together with 0 false negatives and 0 false positives
Boston Housing Data: Application “$p < n$”

Predicting median price of homes in Boston on the basis of 13 predictors, $n = 506$

(1) Conjugate SSVS Gibbs sampler (CPU 21s)
\[ \nu_0 = 0.01, \nu_1 = 1000, \theta \sim B(1, 1), \text{10 000 iterations} \]
Median probability model includes \{5, 6, 8, 11, 13\}

(2) Exhaustive evaluation of posterior model probabilities (CPU 6s)
\[ \nu_0 = 0, \nu_1 = 1000 \text{ and } \theta \sim B(1, 1) \]
(3) EMVS algorithm (CPU 0.53s)

\[ v_1 = 1000, \quad v_0 \in \{10^{-6} + k \times 0.001; 1 \leq k \leq 50\}, \quad \theta \sim B(1, 1) \]

\[ \beta^{(0)} \sim N_{13}(0, 10 \times I_{13}), \quad \text{best model contains \{5, 6, 8, 11, 13\}} \]
Spellman (1998) describe a yeast experiment to identify TF binding sites associated with cell cycle.

800 genes found to have a periodic expression pattern across 2 cell cycles.

Another 800 that do not show any differential pattern across time selected as a reference.

Response vector $\mathbf{Y} = (Y_1, \ldots, Y_{1600})'$ summarizes expression of 1600 genes over time.

Can we explain the gene expression pattern by the occurrence of shared regulatory motifs?
Detecting DNA Binding Motifs Using EMVS

- Promoter regions of each gene screened for DNA motifs
- Motif ≡ word of length 7 consisting of letters \{A, G, T, C\} (altogether \(4^7/2 = 8,192\) motifs)
- Regression matrix \(X_{1600 \times 8192}\) contains counts of occurrences of each motif in the promoter region of each gene
- The motifs lie on a network with similar motifs being the neighbors
- For instance, \(ACCTGTC\) and \(TCCTGTC\) differ by only one letter
  - they are connected on a graph
- Similar motifs assumed to attract the same TFs imply influence gene expression in a similar way
(a) Beta-binomial Model

\[ \nu_0 \in \{0.001 + k \times 1 : 0 \leq k \leq 20\} \]

\[ \nu_1 : \text{random in EM with } a_{\nu_1} = 0.5, b_{\nu_1} = 250, \text{ fixed to 1000 in } g_0(\cdot) \]

\[ \beta^{(0)} \text{ according to (5) with } \nu_0 = 1 \text{ and } \nu_1 = 1000 \]
(b) MRF Model

\[ \nu_0 \in \{10^{-5} + k \times 10^{-5} : 0 \leq k \leq 30\} \]

\[ \nu_1 \text{ and } \beta^0 \text{ as in (a)} \]

\[ \text{Prior } \pi(\theta) \text{ located in the phase transition region} \]
## Results

<table>
<thead>
<tr>
<th>18 Selected Motifs</th>
<th>7 Selected Motifs</th>
<th>Known</th>
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<tbody>
<tr>
<td>(a)</td>
<td>(b)</td>
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<tr>
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<td>TTTATCG</td>
</tr>
</tbody>
</table>

Known motifs found in the SCPD (Sacharomyces Cerevisiae Pomoter Database)
We develop a rapid deterministic method based on EM algorithm as an alternative to stochastic model search.

Regularization diagram combined with rigorous model evaluation enable simultaneous exploration and evaluation of candidate models.

EMVS framework encompasses situations with structured covariates.

Heavy-tailed slab distributions can be considered to alleviate over-shrinkage.

Extensions to multivariate/factor analytic models possible (Rockova and Lesaffre (2013)).
Thank you!
References

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Bayesian Variable Selection in Structured High-dimensional Covariate Spaces with Applications in Genomics

Bayesian Sparse Factor Regression Approach to Genomic Data Integration
To appear in Proceedings of the 28th IWSM

Incorporating Grouping in Bayesian Variable Selection with Applications in Genomics
Under revision for Bayesian Analysis

A Bayesian Graphical Modeling Approach to MicroRNA Regulatory Networks
Annals of Applied Statistics (4) 2024-2048
EMVS Algorithm: a Closer Look

Objective function:

\[
Q\left(\beta, \theta, \sigma \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}\right) = C(\gamma) + Q_1\left(\beta, \sigma \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}\right) \\
+ Q_2\left(\theta \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}\right),
\]

where

\[
Q_1\left(\beta, \sigma \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}\right) = -\frac{(Y - X\beta)'(Y - X\beta)}{2\sigma^2} - \frac{n + p + \nu}{2} \log(\sigma^2) - \frac{\nu \lambda}{2\sigma^2} \\
- \frac{1}{2\sigma^2} \sum_{i=1}^{p} \beta_i^2 E_{\gamma} \cdot \left[\frac{1}{\nu_0(1 - \gamma_i) + \nu_1\gamma_i}\right],
\]

\[
Q_2\left(\theta \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}\right) = E_{\gamma} \cdot \log \pi(\gamma \mid \theta) + \log \pi(\theta),
\]

and \(E_{\gamma} \cdot (\cdot)\) denotes the conditional expectation \(E_{\gamma \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, y(\cdot)}\).
EMVS Algorithm (Beta-binomial Case)

Objective function:

\[ Q(\beta, \theta, \sigma \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}) = C(\gamma) + Q_1(\beta, \sigma \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}) + Q_2(\theta \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}) \]

where

\[ Q_1(\beta, \sigma \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}) = -\frac{(Y - X\beta)'(Y - X\beta)}{2\sigma^2} - \frac{n + p + \nu}{2} \log(\sigma^2) - \frac{\nu \lambda}{2\sigma^2} \]

\[ -\frac{1}{2\sigma^2} \sum_{i=1}^{p} \beta_i^2 \mathbb{E}_{\gamma} \left[ \frac{1}{v_0(1 - \gamma_i) + v_1 \gamma_i} \right] \]

\[ Q_2(\theta \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}) = \sum_{i=1}^{p} \log \left( \frac{\theta}{1 - \theta} \right) \mathbb{E}_{\gamma_i} \gamma_i + (a - 1) \log \theta + (b + p - 1) \log(1 - \theta), \]

and \( \mathbb{E}_{\gamma_i}(\cdot) \) denotes the conditional expectation \( \mathbb{E}_{\gamma_i \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, \gamma} \).
Variables $\gamma$ depend on the data $Y$ only through the current estimates $\beta^{(k)}$.

We have:

\begin{equation}
\mathbb{E}_{\gamma \mid \gamma_i} = P(\gamma_i = 1 \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, Y) = P(\gamma_i = 1 \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}) = p_i^*,
\end{equation}

where

\begin{align*}
p_i^* &= \frac{\pi(\beta_i^{(k)} \mid \sigma^{(k)}, \gamma_i = 1)P(\gamma_i = 1 \mid \theta^{(k)})}{\pi(\beta_i^{(k)} \mid \sigma^{(k)}, \gamma_i = 1)P(\gamma_i = 1 \mid \theta^{(k)}) + \pi(\beta_i^{(k)} \mid \sigma^{(k)}, \gamma_i = 0)P(\gamma_i = 0 \mid \theta^{(k)})}
\end{align*}

are the mixing proportions when fitting a Gaussian mixture model via EM.

\begin{equation}
\mathbb{E}_{\gamma \mid \cdot} \left[ \frac{1}{v_0(1 - \gamma_i) + v_1 \gamma_i} \right] = \frac{\mathbb{E}_{\gamma \mid \cdot} (1 - \gamma_i)}{v_0} + \frac{\mathbb{E}_{\gamma \mid \cdot} \gamma_i}{v_1} = \frac{1 - p_i^*}{v_0} + \frac{p_i^*}{v_1} = d_i^*
\end{equation}
M-step (Beta-binomial Case)

(1) Update $\beta^{(k+1)}$ Closed form ridge regression solution

$$\beta^{(k+1)} = (X'X + D^*)^{-1}X'Y, \quad D^* = \text{diag}\{d_1^*, \ldots, d_p^*\}$$

$\Rightarrow$ For $p > n$, use Woodbury-Sherman formula to get

$$\beta^{(k+1)} = \left[D^*^{-1} - D^*^{-1}X'\left(I_{n\times n} + XD^*^{-1}X'\right)^{-1}XD^*^{-1}\right]X'y$$

(2) Update $\sigma^{(k+1)}$ Closed form

$$\sigma^{(k+1)} = \sqrt{\frac{|Y - X\beta^{(k+1)}|_2^2 + |D^{1/2}\beta^{(k+1)}|_2^2}{n + p + \eta}}.$$

(2) Update $\theta^{(k+1)}$ Closed form

$$\theta^{(k+1)} = \frac{\sum_{i=1}^p p_i^* + a - 1}{a + b + p - 2}.$$
EMVS for Structured Priors

(a) *Logistic regression product prior*

\[
Q_2 \left( \theta | \beta^{(k)}, \Theta^{(k)}, \sigma^{(k)} \right) = \sum_{i=1}^{p} \{ Z_i' \theta E_{\gamma_i} \gamma_i - \log[1 + \exp(Z_i' \theta)] \} + \sum_{j=1}^{q} \log \pi(\theta_j),
\]

Beta distribution on the inverse logistic transformation of \( \theta_j \)

\[
\pi(\theta_j) = \frac{1}{B(a, b)} \left[ \frac{\exp(\theta_j)}{1 + \exp(\theta_j)} \right]^a \left[ \frac{1}{1 + \exp(\theta_j)} \right]^b \] (3)

(b) *MRF prior with \( \theta_1 = \theta(1, \ldots, 1)' \), where \( \theta \sim \pi(\theta) \) in (3)*

\[
Q_2(\theta | \beta^{(k)}, \Theta^{(k)}, \sigma^{(k)}) = \theta \left( \sum_{i=1}^{p} E_{\gamma_i} \gamma_i + a \right) + \psi(\theta, \theta_2) - (a + b) \log[1 + \exp(\theta)].
\]

\( \sim E_{\gamma_i} \gamma_i \) complicated due to dependence between \( \gamma_i \)'s

\( \sim \psi(\theta, \theta_2) \) not in closed form
EMVS for Structured Priors

(a) **Logistic regression product prior**

\[
Q_2 \left( \theta \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)} \right) = \sum_{i=1}^{p} \left\{ Z_i \theta \cdot E_{\gamma_i} - \log[1 + \exp(Z_i \theta)] \right\} + \sum_{j=1}^{q} \log \pi(\theta_j),
\]

Beta distribution on the inverse logistic transformation of \( \theta_j \)

\[
\pi(\theta_j) = \frac{1}{B(a, b)} \left[ \frac{\exp(\theta_j)}{1 + \exp(\theta_j)} \right]^a \left[ \frac{1}{1 + \exp(\theta_j)} \right]^b
\]

(b) **MRF prior with \( \theta_1 = \theta(1, \ldots, 1)' \), where \( \theta \sim \pi(\theta) \) in (3)**

\[
Q_2(\theta \mid \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}) = \theta \left( \sum_{i=1}^{p} E_{\gamma_i} + a \right) + \psi(\theta, \theta_2) - (a + b) \log[1 + \exp(\theta)].
\]

\[
E_{\gamma_i} \text{ complicated due to dependence between } \gamma_i's
\]

\[
\psi(\theta, \theta_2) \text{ not in closed form}
\]
Approximated E-step (MRF Prior)

Conditional posterior distribution \( \pi(\gamma | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, y) \) proportional to

\[
\exp \left[ \left( \frac{1}{2} \log \left( \frac{v_0}{v_1} \right) \mathbf{1}' - \frac{v_0 - v_1}{2\sigma(k)^2 v_1 v_0} \beta^{(k)'} \text{diag}\{\beta^{(k)}\}_{i=1}^{p} + \theta^{(k)} \mathbf{1}' \right) \gamma + \gamma' \theta_2 \gamma \right].
\]

Markov random field distribution \( \text{MRF}(\theta^*, \theta_2) \)

\[\Rightarrow\] Expectation \( E(\gamma | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, y) = \frac{\partial \psi(\theta^*, \theta_2)}{\partial \theta} \bigg|_{\theta = \theta^*} \) not analytically tractable

\[\Rightarrow\] Mean field approximation \( \Rightarrow \) iteratively solving

\[
\hat{\mu}_i = \frac{\exp(\theta^*_i + \sum_{j \neq i} \theta_{ij} \hat{\mu}_j)}{1 + \exp(\theta^*_i + \sum_{j \neq i} \theta_{ij} \hat{\mu}_j)}, \quad 1 \leq i \leq p
\]

\[\Rightarrow\] We obtain approximations to the mixing proportions

\[
\hat{\mu}_i = P(\gamma_i = 1 | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, y)
\]
Approximated E-step (MRF Prior)

Conditional posterior distribution \( \pi(\gamma | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, y) \) proportional to

\[
\exp \left[ \left( \frac{1}{2} \log \left( \frac{v_0}{v_1} \right) \right) \mathbf{1}' - \frac{v_0 - v_1}{2\sigma^{(k)^2} v_1 v_0} \beta^{(k)}' \text{diag} \{ \beta^{(k)}_i \}_{i=1}^p + \theta^{(k)} \mathbf{1}' \right] \gamma + \gamma' \theta_2 \gamma \right].
\]

Markov random field distribution \( \text{MRF}\left(\theta^*, \theta_2\right) \)

\( \leadsto \) Expectation \( \mathbb{E}(\gamma | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, y) = \frac{\partial \psi(\theta_1, \theta_2)}{\partial \theta_1} |_{\theta=\theta^*} \) not analytically tractable

\( \leadsto \) Mean field approximation \( \leadsto \) iteratively solving

\[
\hat{\mu}_i = \frac{\exp(\theta^*_i + \sum_{j \neq i} \theta_{ij} \hat{\mu}_j)}{1 + \exp(\theta^*_i + \sum_{j \neq i} \theta_{ij} \hat{\mu}_j)}, \quad 1 \leq i \leq p
\]

\( \leadsto \) We obtain approximations to the mixing proportions

\[
\hat{\mu}_i = \mathbb{P}(\gamma_i = 1 | \beta^{(k)}, \theta^{(k)}, \sigma^{(k)}, y)
\]
Approximated M-step (MRF Prior)

\[ \Rightarrow \text{Updates for } \beta \text{ and } \sigma \text{ the same as before} \]

\[ \Rightarrow \text{Update } \theta \]

\[ \theta^{(k+1)} = \arg\max_{\theta \in \mathbb{R}} \left\{ \theta \left( \sum_{i=1}^{p} \hat{\mu}_i^* + a \right) + \psi(\theta, \theta_2) - (a + b) \log[1 + \exp(\theta)] \right\} . \]

Mean field approximation to the partition function \( \psi(\theta, \theta_2) \)

\[ \psi(\theta, \theta_2) \approx \theta \sum_{i=1}^{p} \mu_i + \mu' \theta_2 \mu - \psi^*(\mu), \quad (4) \]

where \( \mu_i = \mathbb{E}_{\beta, \theta_2}(\gamma_i) \) and \( \psi^*(\cdot) \) denotes the conjugate dual function

\[ \psi^*(\mu) = \sum_{i=1}^{p} [\mu_i \log \mu_i + (1 - \mu_i) \log(1 - \mu_i)] \]
Deterministic Annealing EM (DAEM)

DAEM algorithm to maximize (2)

(1) Logistic/beta-binomial prior:
Mixing proportions replaced by

\[ p_{i,t}^* = \frac{\pi(\beta_i^{(k)} | \sigma(k), \gamma_i = 1)^t P(\gamma_i = 1 | b^{(k)})^t}{\pi(\beta_i^{(k)} | \sigma(k), \gamma_i = 1)^t P(\gamma_i = 1 | b^{(k)})^t + \pi(\beta_i^{(k)} | \sigma(k), \gamma_i = 0)^t P(\gamma_i = 0 | b^{(k)})^t}, \]

\[ \rightarrow \]

Limiting behavior \( p_{i,t}^* \rightarrow 0.5 \) as \( t \rightarrow 0 \) suggests a starting vector

\[ \hat{\beta}_{t=0} = \left[ X'X + \frac{v_0 + v_1}{2v_0v_1} I_p \right]^{-1} X'y \]  

(2) MRF prior:
MF approximation to evaluate expectation of \( \text{MRF}(t \theta^*, t \theta_2) \).
Phase Transition

\[ \sum_{i=1}^{p} E_{\theta,\theta_2} \gamma_i \]

\[ Q_{2}^{MRF}(\theta) \]

\[ \pi(\theta) \]
Heavy-tailed Alternative

⇝ Gaussian slab density may **overshrink** ⇝ put prior on $v_1$ to induce **heavier tails**

⇝ Prior suggested in the "g-prior" context (Cui and George (2008), Maruyama and George (2011))

$$p(v_1) = \frac{v_1^b(1 + v_1)^{-a-b-2}}{B(a + 1, b + 1)} I_{(0, \infty)}(v_1)$$ (6)

⇝ Implied marginal prior distribution

$$\pi(\beta_i|v_0, \sigma, \gamma) = (1 - \gamma_i)N(0, \sigma^2 v_0) + \gamma_i \tilde{\pi}_{a, b, \sigma}(\beta_i),$$

where

$$\tilde{\pi}_{a, b, \sigma}(\beta_i) \propto \exp\left(\frac{\beta_i^2}{4\sigma^2}\right) \left(\frac{\beta_i^2}{2\sigma^2}\right)^{b - \frac{1}{4}} \sqrt{2\pi\sigma^2} W_{-a - \frac{b}{2} - \frac{5}{4}, -\frac{b}{2} - \frac{1}{4}} \left(\frac{\beta_i^2}{2\sigma^2}\right)$$
Heavy-tailed Alternative

\[ \Rightarrow \text{Integrating out } v_1 \text{ complicates the tractability of the M-step} \]

\[ \Rightarrow \text{We treat } v_1 \text{ as an additional unknown parameter} \]

\[ \Rightarrow \text{Assuming prior (6) we update } v_1 \text{ at the } k\text{-th iteration according to} \]

\[
v_1^{(k+1)} = \arg\max_{v_1} \left\{ -\frac{|P^{*1/2} \beta|_2}{2\sigma^{(k+1)}} \frac{1}{v_1} + \left( b - \sum_{i=1}^{p} \frac{p_i^*}{2} \right) \log(v_1) - (a + b + 2) \log(1 + v_1) \right\},
\]

where \( P^* = \text{diag}\{p_1^*, \ldots, p_p^*\} \)

\[ \Rightarrow \text{EM algorithm remains unchanged, just with updates based on the current value } v_1^{(k)} \]