An introduction to INLA with a comparison to JAGS

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(Thanks to Håvard Rue, University of Science and Technology Trondheim, Norway)

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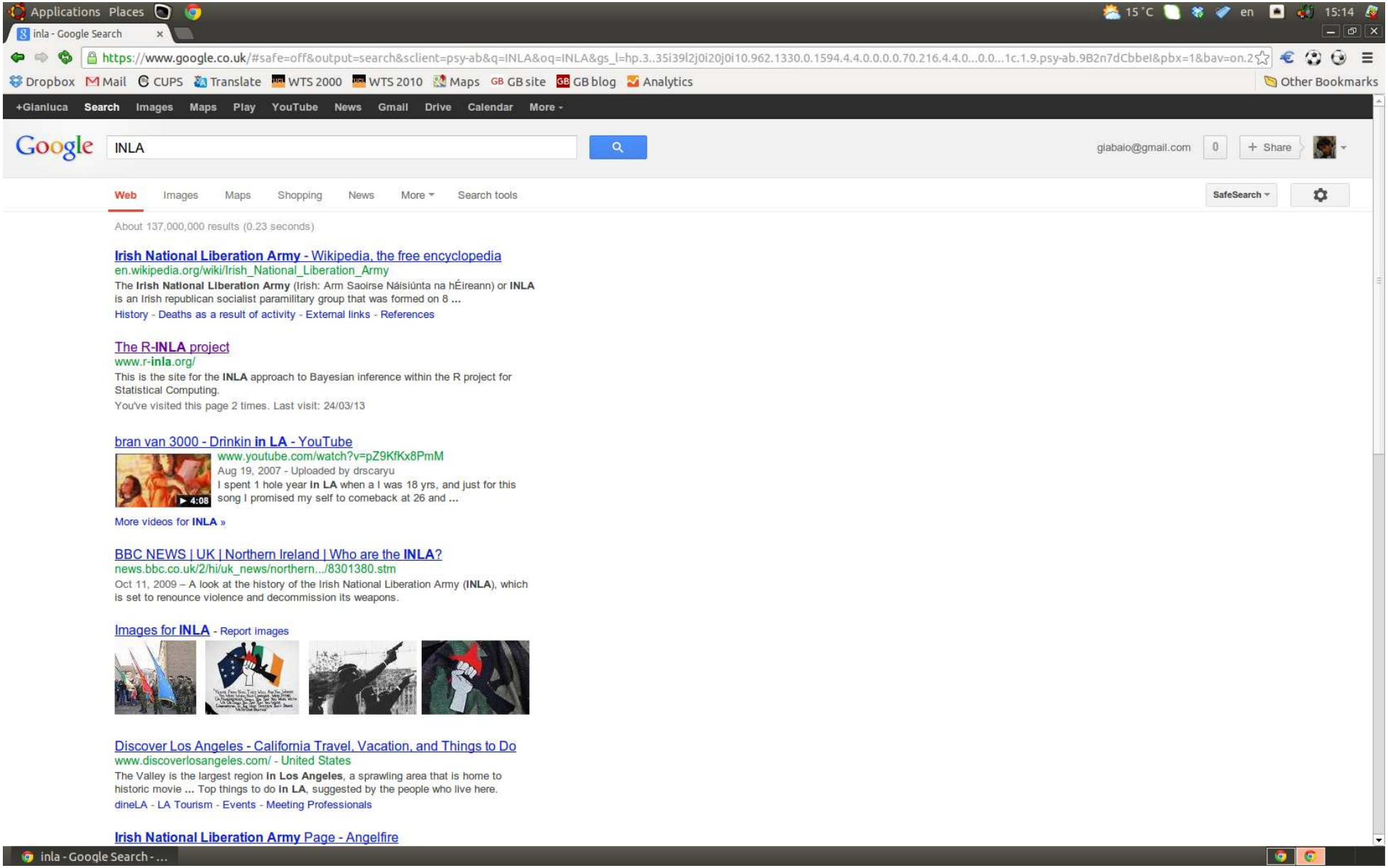
Laplace's liberation army (?)



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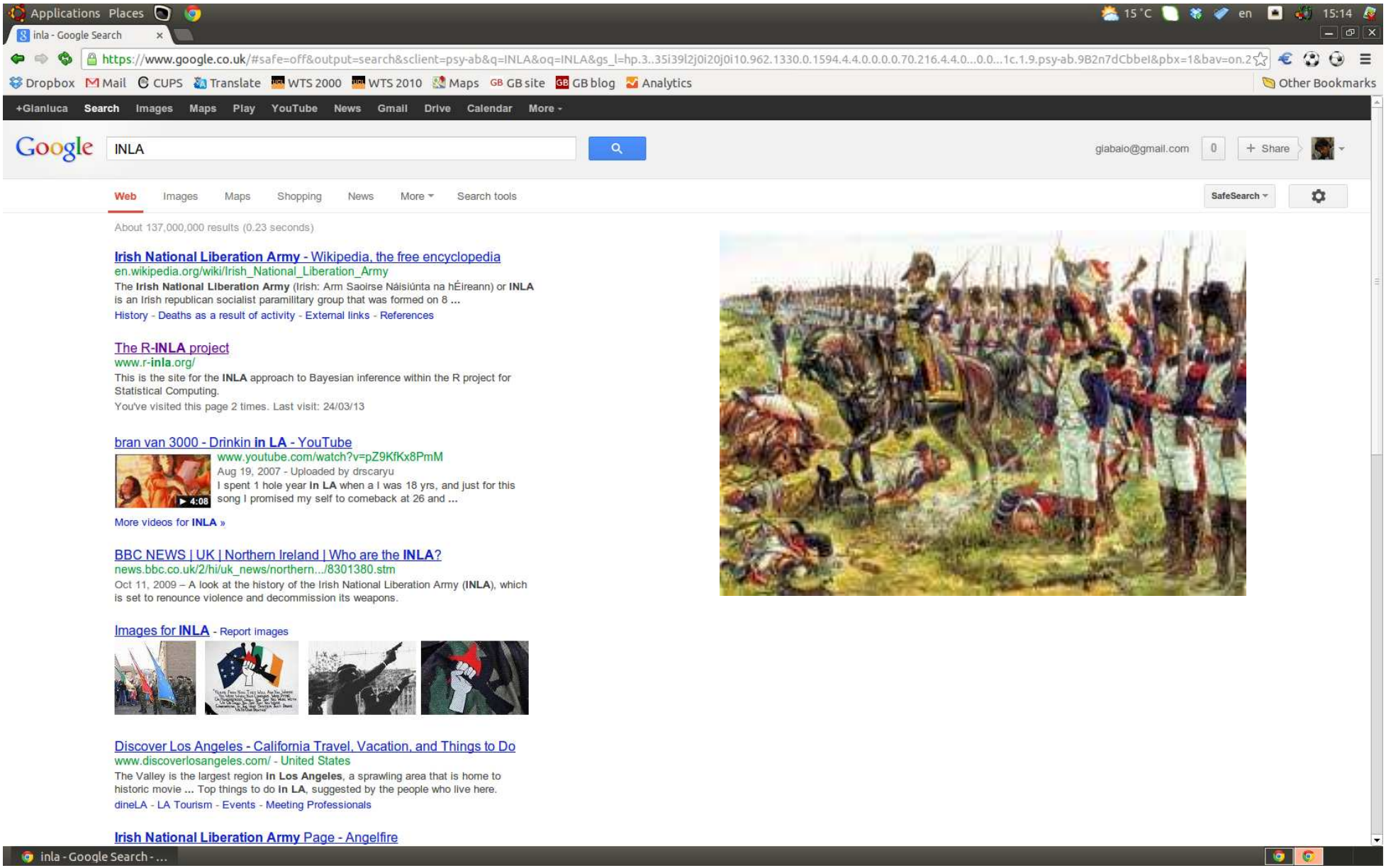

Outline of presentation

1. **9.00 – 9.45: (Quick & moderately clean) introduction to Bayesian computation**
   - MCMC
   - Latent Gaussian models
   - Gaussian Markov Random Fields
2. 9.45 – 10.00: Coffee break
3. 10.00 – 10.45: Introduction to INLA
   - Basic ideas
   - Some details
   - A simple example
4. 10.45 – 11.00: Coffee break
5. 11.00 – 12.00: Using the package `R-INLA`
   - How does it work?
   - Some simple examples
   - (Slightly) more complex examples

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(Quick & moderately clean) introduction to Bayesian computation

Bayesian computation

- In a (**very** small!) nutshell, Bayesian inference boils down to the computation of **posterior** and/or **predictive** distributions

$$p(\boldsymbol{\theta} \mid \boldsymbol{y}) = \frac{p(\boldsymbol{y} \mid \boldsymbol{\theta})p(\boldsymbol{\theta})}{\int p(\boldsymbol{y} \mid \boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}} \qquad p(y^* \mid \boldsymbol{y}) = \int p(y^* \mid \boldsymbol{\theta})p(\boldsymbol{\theta} \mid \boldsymbol{y})d\boldsymbol{\theta}$$

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- Since the advent of simulation-based techniques (notably MCMC), Bayesian computation has enjoyed incredible development
- This has certainly been helped by dedicated software (eg `BUGS` and then `WinBUGS`, `OpenBUGS`, `JAGS`)
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- MCMC methods are very general and can effectively be applied to "any" model
- However:
  - Even if **in theory**, MCMC can provide (nearly) exact inference, given perfect convergence and MC error $\to 0$, in practice, this has to be balanced with model complexity and running time
  - This is particularly an issue for problems characterised by large data or very complex structure (eg hierarchical models)

MCMC — Gibbs sampling

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  3. Repeat step 2. for S times until convergence is reached to the target distribution $p(\boldsymbol{\theta} \mid y)$
  4. Use the sample from the target distribution to compute all relevant statistics: (posterior) mean, variance, credibility intervals, etc.

- If the *full conditionals* are not readily available, they need to be estimated (eg via Metropolis-Hastings or slice sampling) before applying the GS

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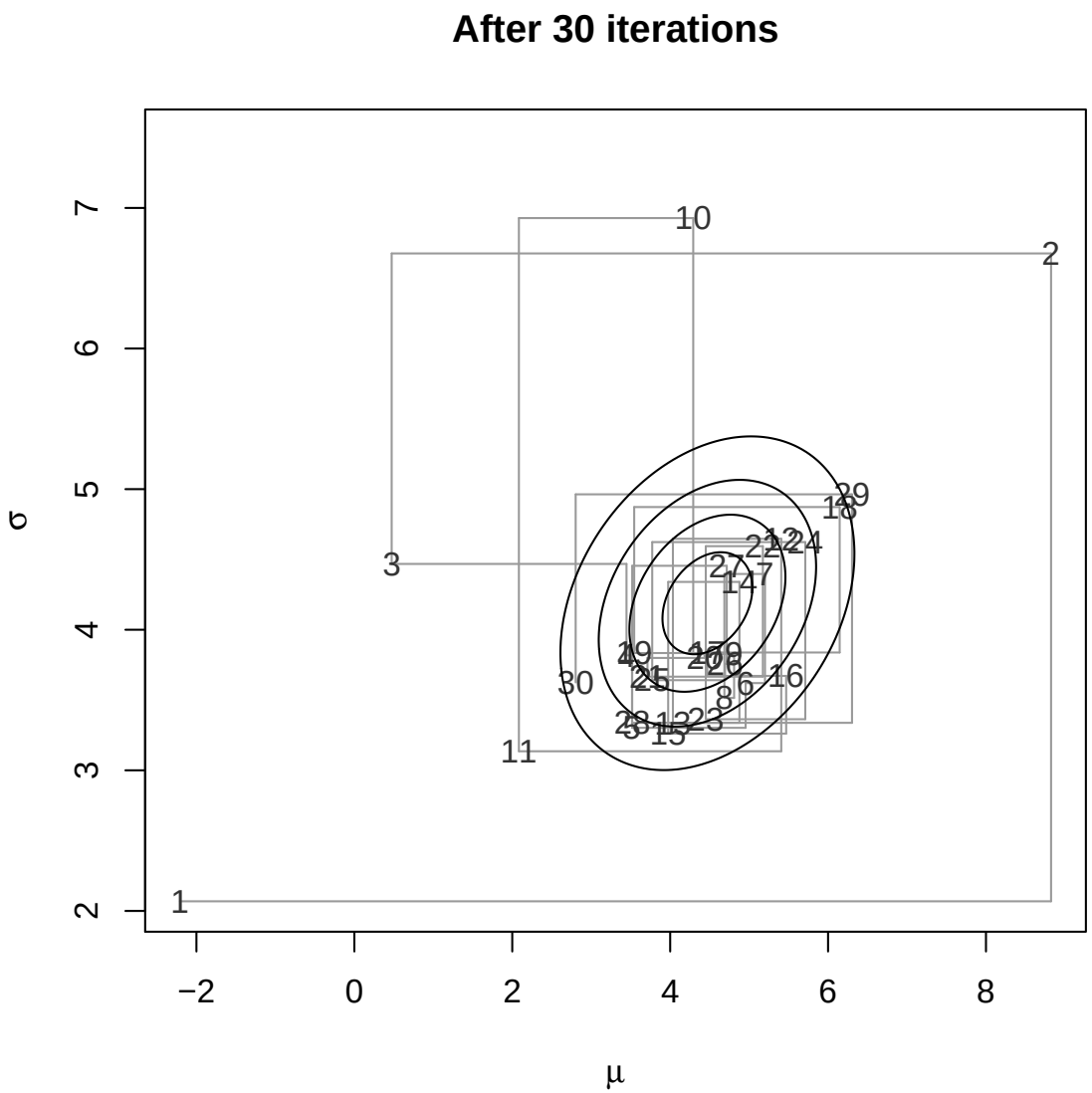

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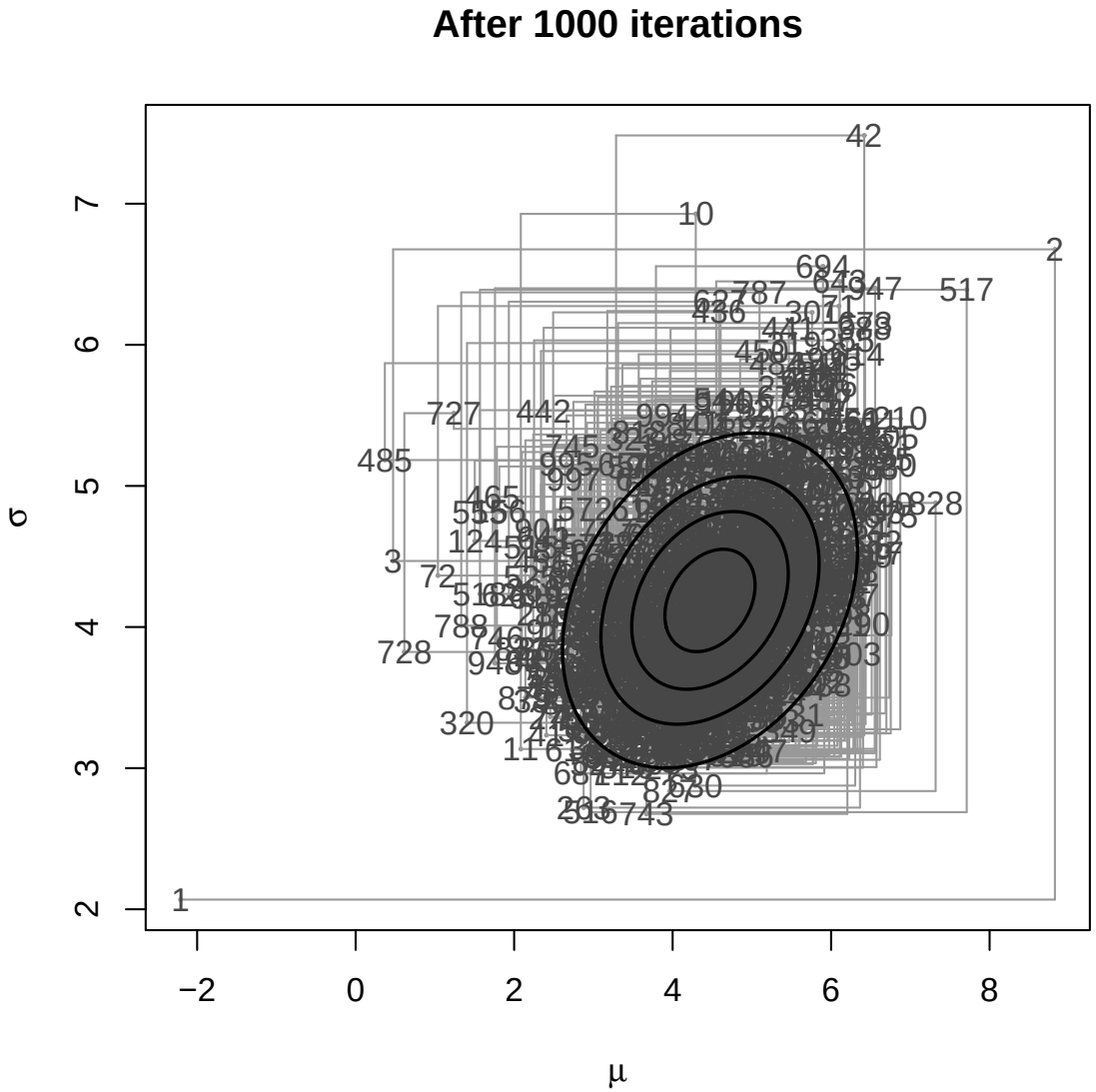

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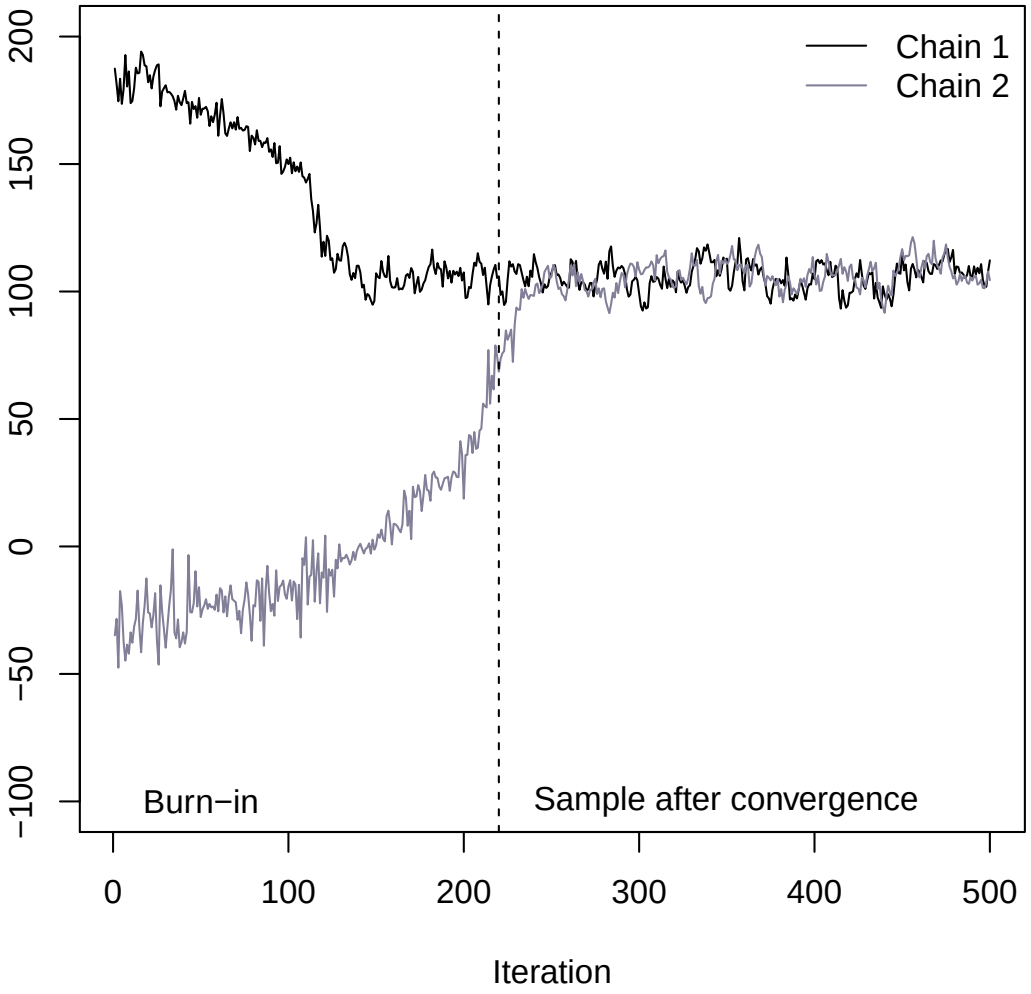

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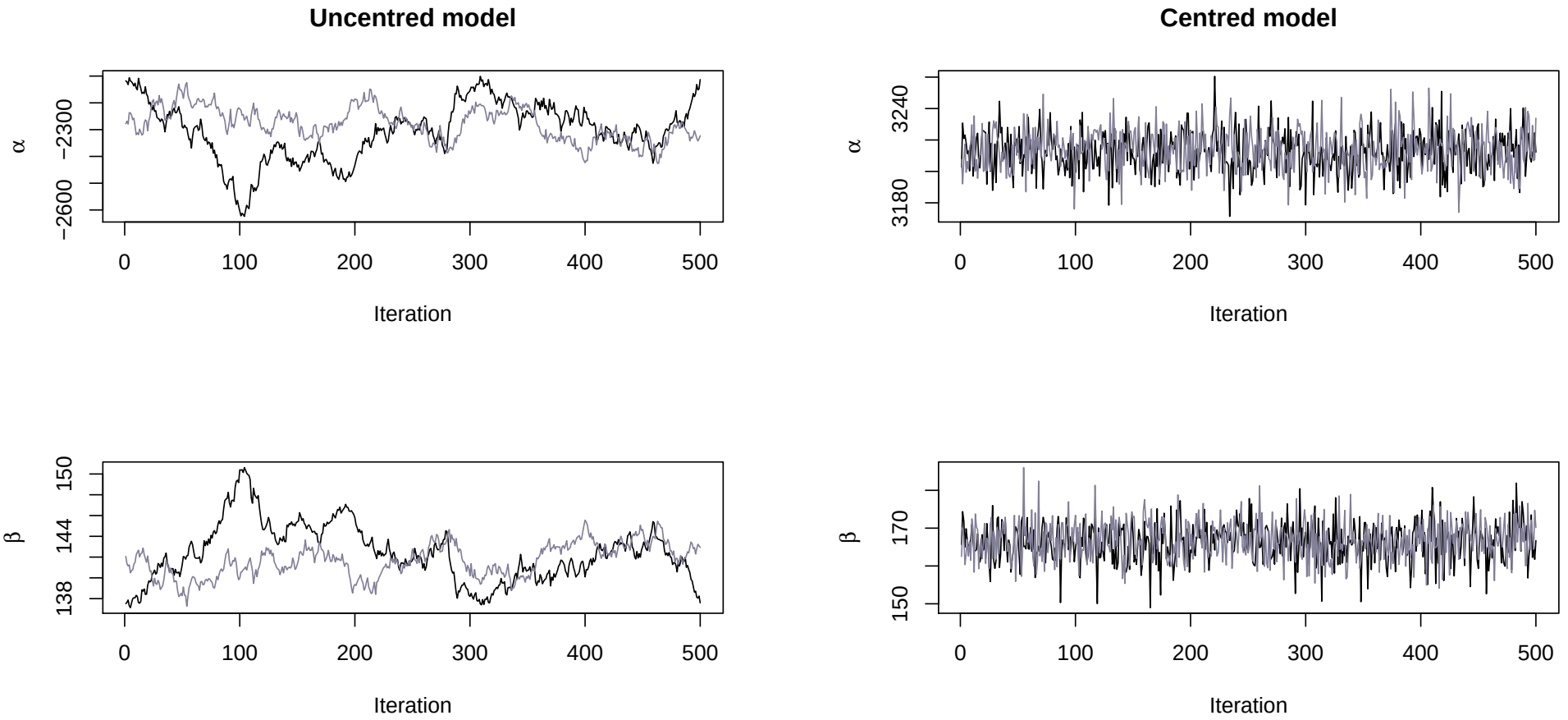
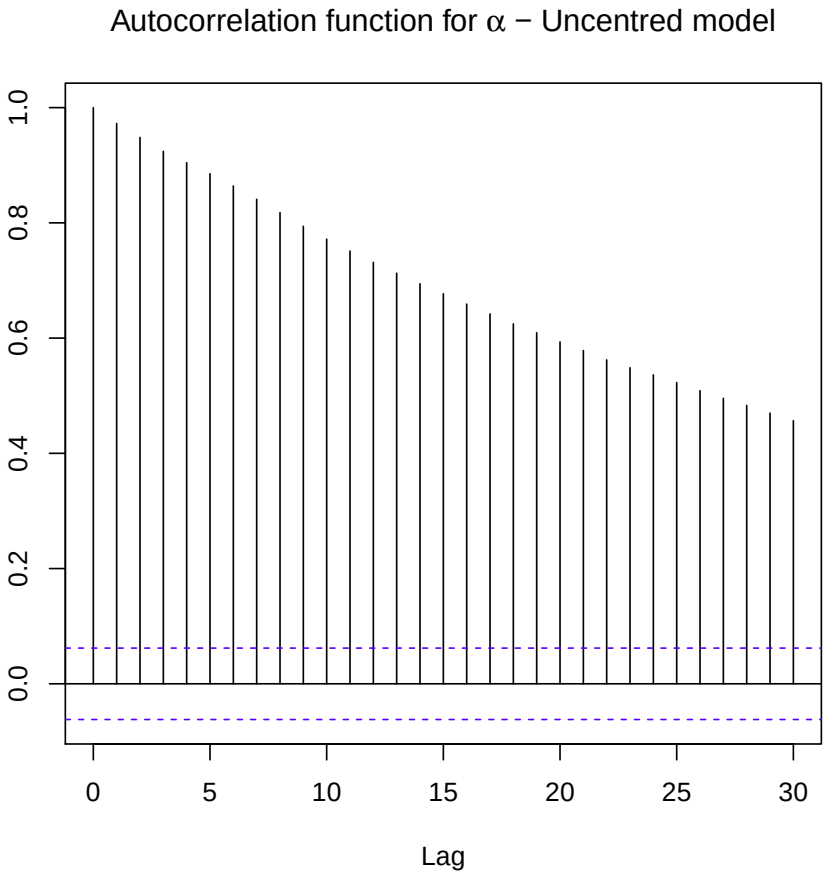
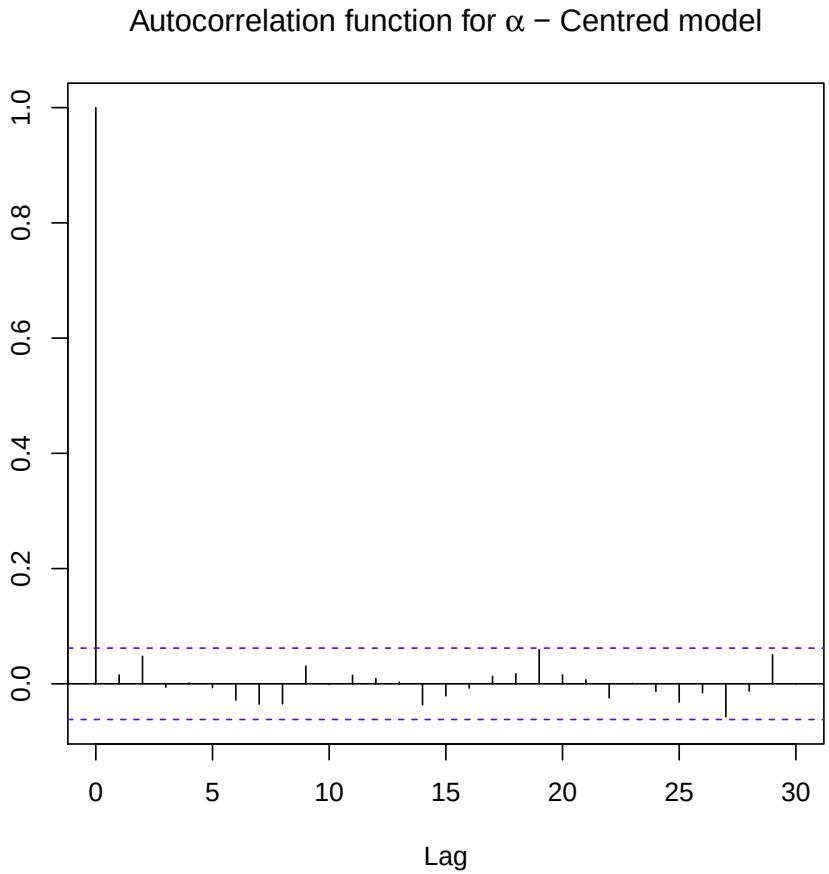


- Formal assessment of convergence: potential scale reduction

$$\hat{R} = \sqrt{\frac{\widehat{\mathsf{Var}}(\theta_k \mid \boldsymbol{y})}{W(\theta_k)}}$$

MCMC — autocorrelation

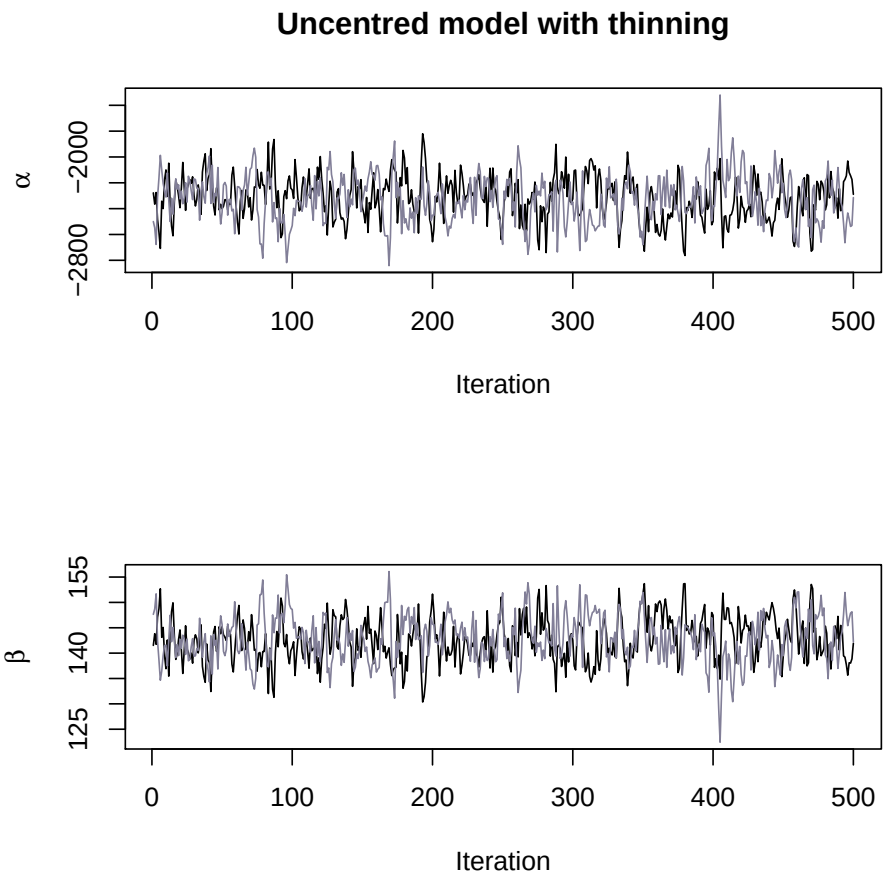


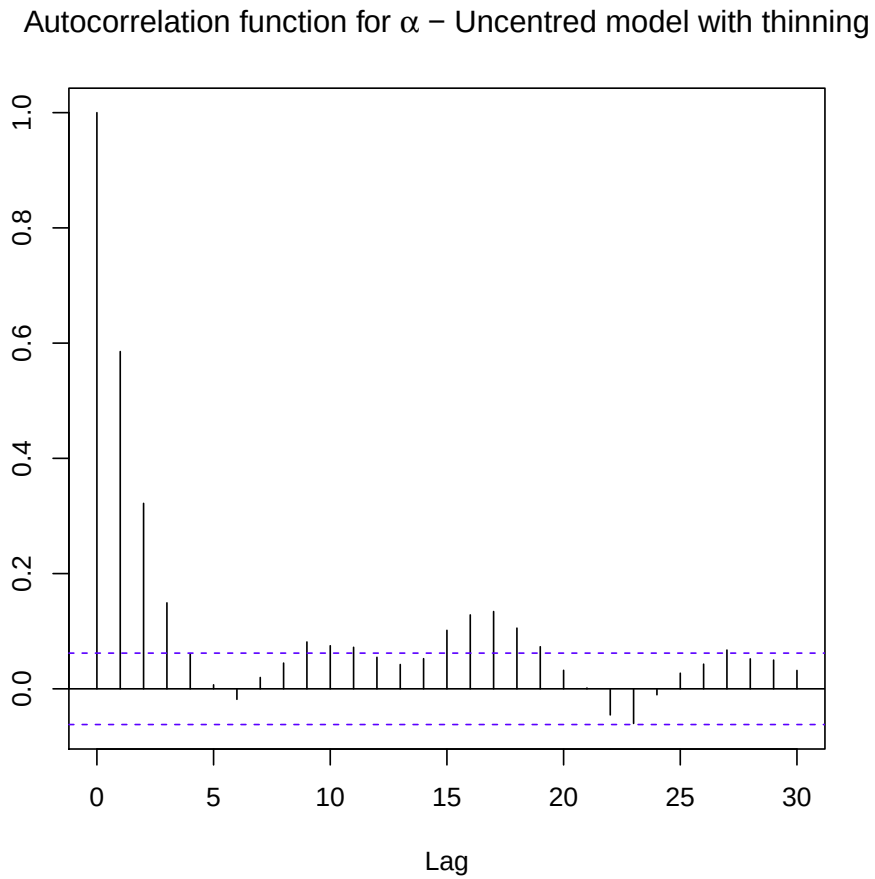


- Formal assessment of autocorrelation: effective sample size

$$n_{\text{eff}} = \frac{S}{1 + 2\sum_{t=1}^{\infty} \rho_t}$$

MCMC — brute force





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- Formulating the model using a specific characterisation
  - All models that can be formulated in this way have certain features in common, which facilitate the computational aspects
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- Compute the relevant quantities typically using numerical methods

Latent Gaussian models (LGMs)

- The general problem of (parametric) inference is posited by assuming a probability model for the observed data, as a function of some relevant parameters

$$\boldsymbol{y} \mid \boldsymbol{\theta}, \boldsymbol{\psi} \sim p(\boldsymbol{y} \mid \boldsymbol{\theta}, \boldsymbol{\psi}) = \prod_{i=1}^{n} p(y_i \mid \boldsymbol{\theta}, \boldsymbol{\psi})$$

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$$\boldsymbol{\theta} \mid \boldsymbol{\psi} \sim \text{Normal}(\mathbf{0}, \boldsymbol{\Sigma}(\boldsymbol{\psi}))$$
$$\theta_l \perp\!\!\!\perp \theta_m \mid \boldsymbol{\theta}_{-lm}$$

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- This kind of models is often referred to as **Latent Gaussian models**

LGMs as a general framework

- In general, we can partition $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ and re-express a LGM as

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\boldsymbol{\psi} &\sim p(\boldsymbol{\psi}) && \text{(“hyperprior”)} \\
\boldsymbol{\theta} \mid \boldsymbol{\psi} &\sim p(\boldsymbol{\theta} \mid \boldsymbol{\psi}) = \text{Normal}(0, \boldsymbol{\Sigma}(\boldsymbol{\psi}_1)) && \text{(“GMRF prior”)} \\
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- The dimension of $\boldsymbol{\theta}$ can be very large (eg 10^2-10^5)
- Conversely, because of the conditional independence properties, the dimension of $\boldsymbol{\psi}$ is generally small (eg 1-5)

LGMs as a general framework

- A very general way of specifying the problem is by modelling the mean for the i-th unit by means of an additive linear predictor, defined on a suitable scale (e.g. logistic for binomial data)

$$\eta_i = \alpha + \sum_{m=1}^{M} \beta_m x_{mi} + \sum_{l=1}^{L} f_l(z_{li})$$

where

  - α is the intercept;
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- **NB**: This of course implies some form of Normally-distributed marginals for $\alpha, \boldsymbol{\beta}$ and $\boldsymbol{f}$

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- Survival models / logGaussian Cox Processes
  - More complex specification in theory, but relatively easy to fit within the INLA framework!

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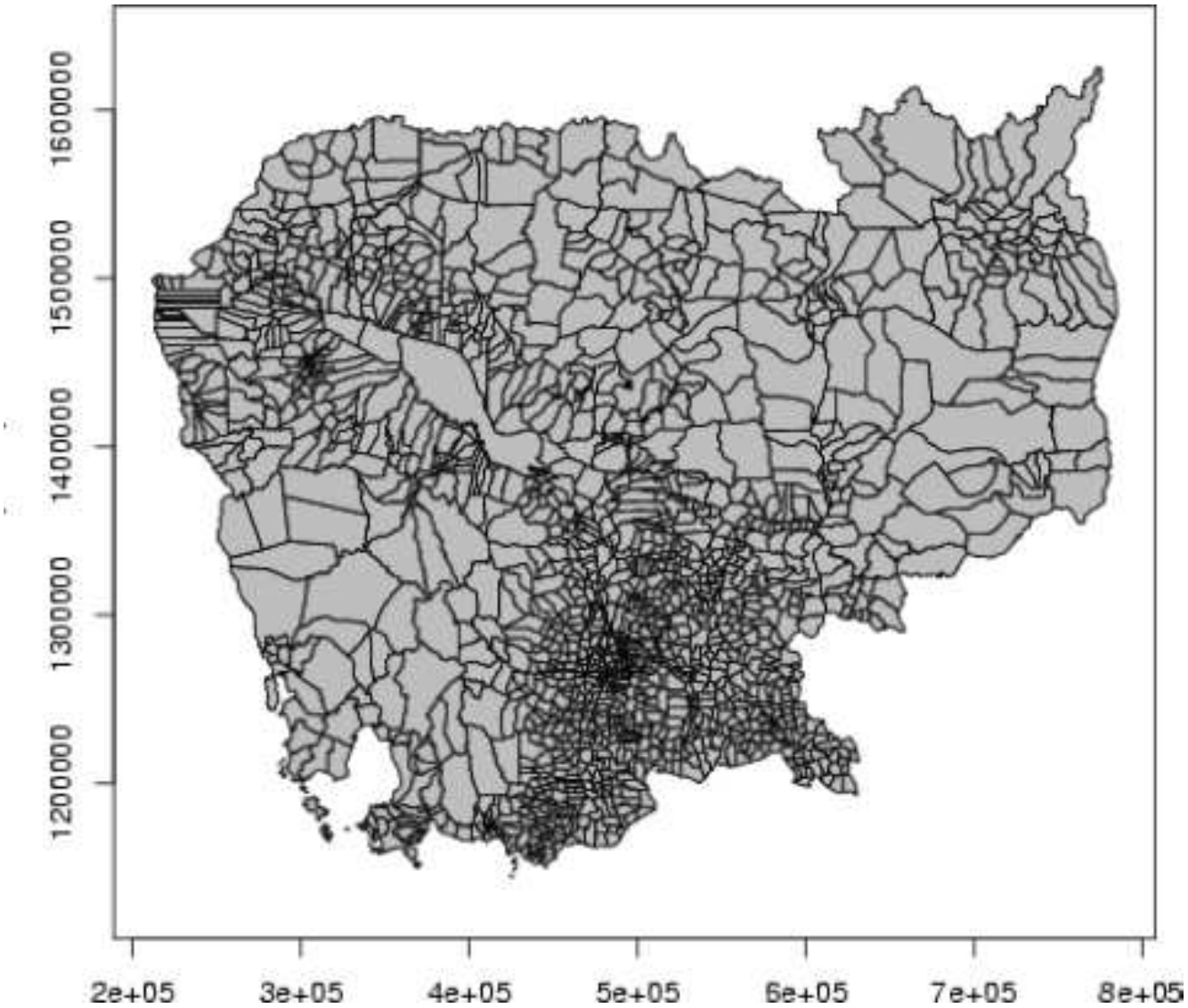
- Conversely, it is much simpler when using the precision matrix $\boldsymbol{Q} =: \boldsymbol{\Sigma}^{-1}$
  - As it turns out, it can be shown that

$$\theta_l \perp\!\!\!\perp \theta_m \mid \boldsymbol{\theta}_{-lm} \Leftrightarrow \boldsymbol{Q}_{lm} = 0$$

  - Thus, under **conditional** independence (which is a less restrictive constraint), the precision matrix is typically **sparse**
  - We can use existing numerical methods to deal with sparse matrices (eg the `R` package `Matrix`)
  - Most computations in GMRFs are performed in terms of computing Cholesky's factorisations

Precision matrix and conditional independence



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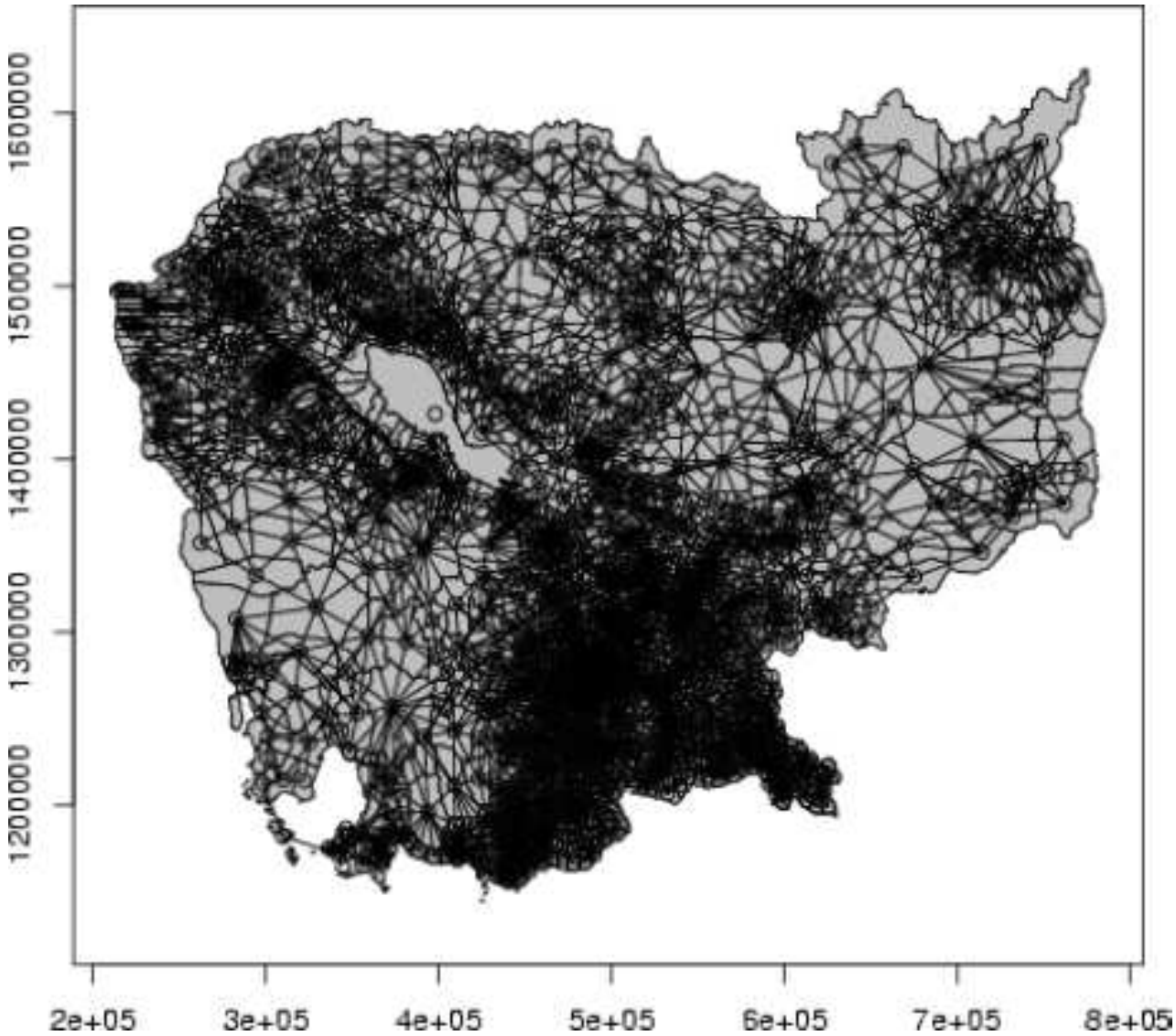

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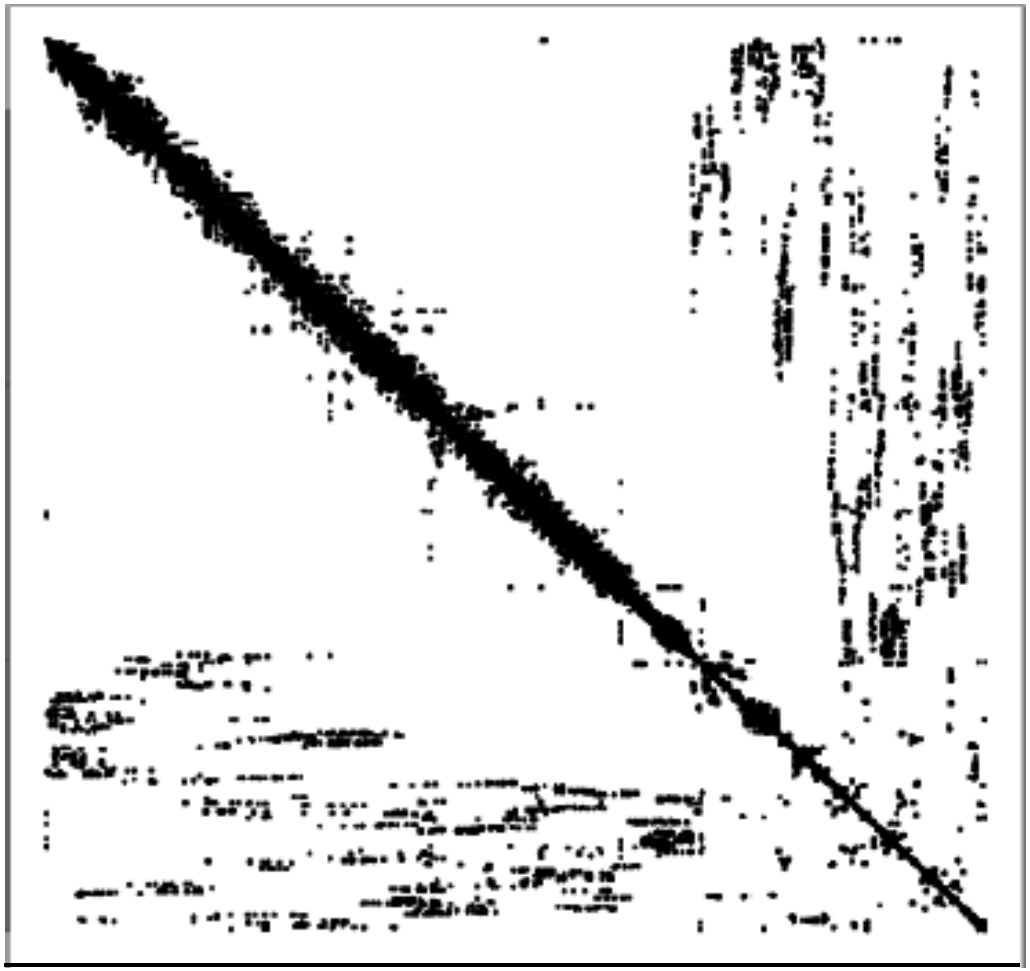

MCMC and LGMs

- (Standard) MCMC methods tend to perform poorly when applied to (non-trivial) LGMs. This is due to several factors
  - The components of the latent Gaussian field $\boldsymbol{\theta}$ tend to be highly correlated, thus impacting on convergence and autocorrelation
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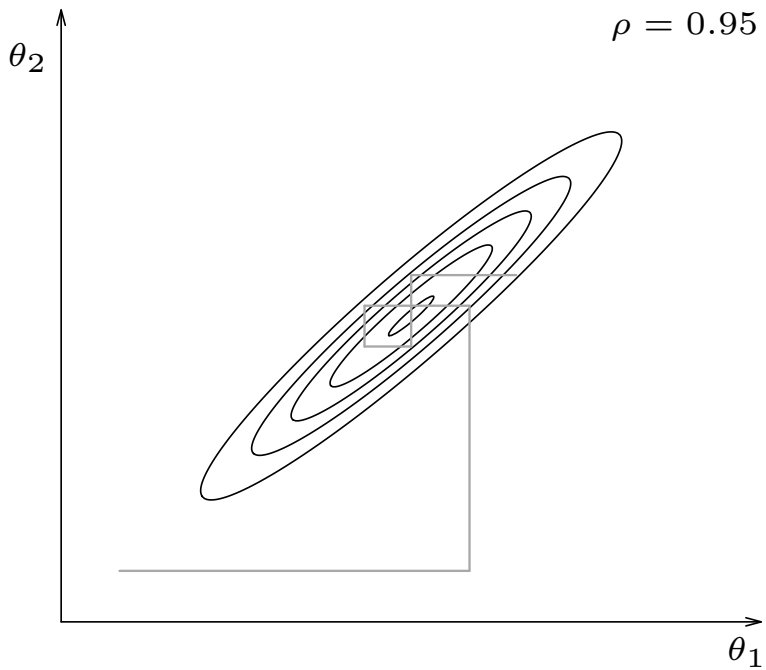

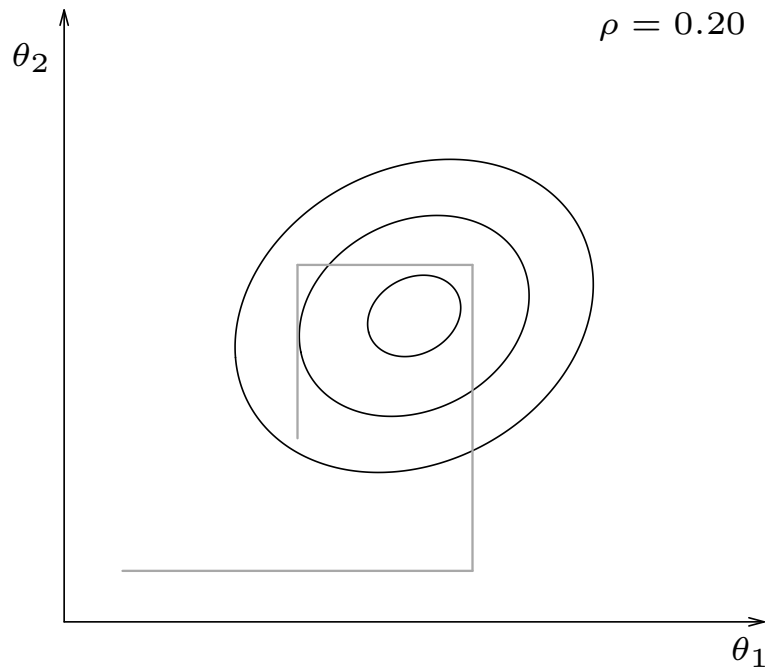

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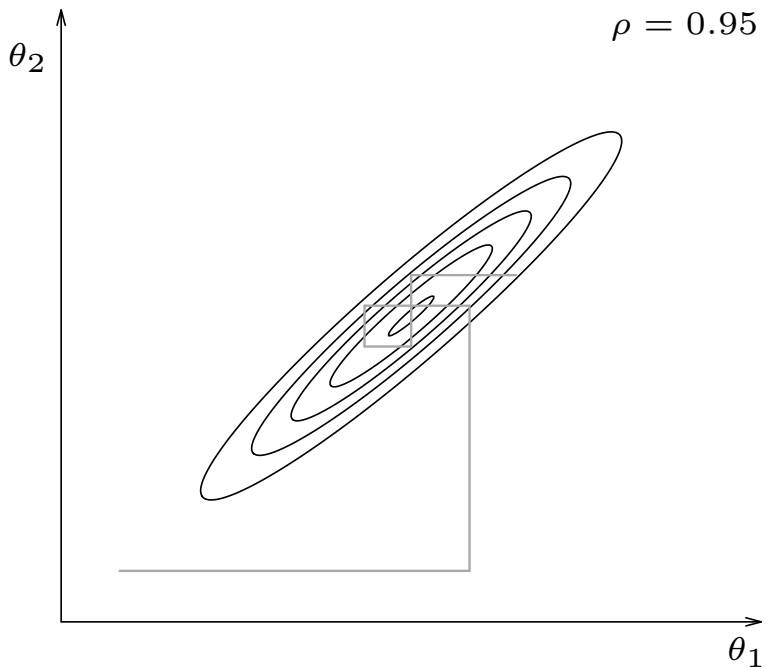

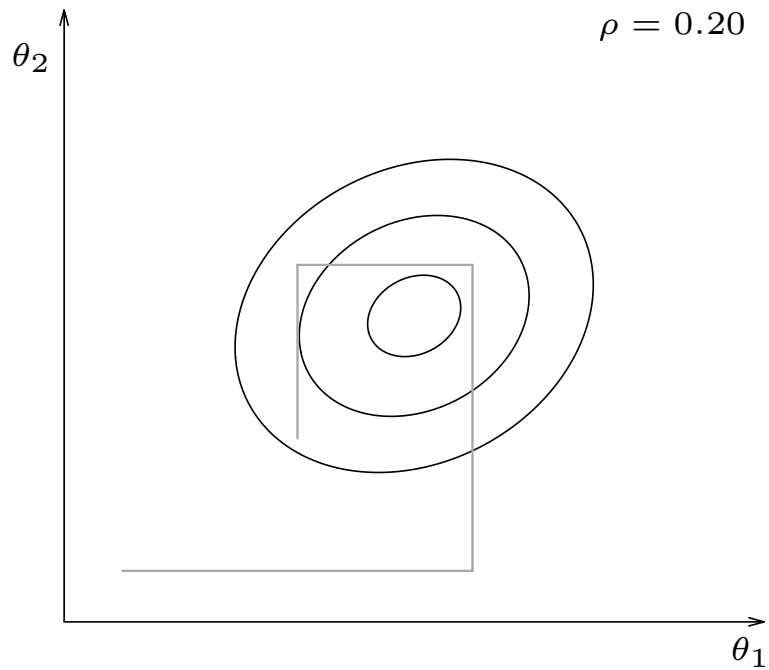

- Again, blocking and overparameterisation can **alleviate**, but rarely eliminate the problem

Summary so far

- Bayesian computation (especially for LGMs) is in general difficult
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- A wide class of statistical models can be represented in terms of LGM
- This allows us to take advantage of nice computational properties
  - GMRFs
  - Sparse precision matrices

- This is at the heart of the INLA approach

Introduction to INLA

- The starting point to understand the INLA approach is the definition of conditional probability, which holds for any pair of variables (x, z) — and, technically, provided $p(z) > 0$

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- In particular, a conditional version can be obtained further considering a third variable w as

$$p(z \mid w) = \frac{p(x, z \mid w)}{p(x \mid z, w)}$$

which is particularly relevant to the Bayesian case

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- Thus, under LA, $g(x) \approx \text{Normal}(\hat{x}, \hat{\sigma}^2)$

Laplace approximation — example

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- Consequently, we can approximate $p(x)$ as

$$p(x) \approx \tilde{p}(x) = \text{Normal}(k-2, 2(k-2))$$

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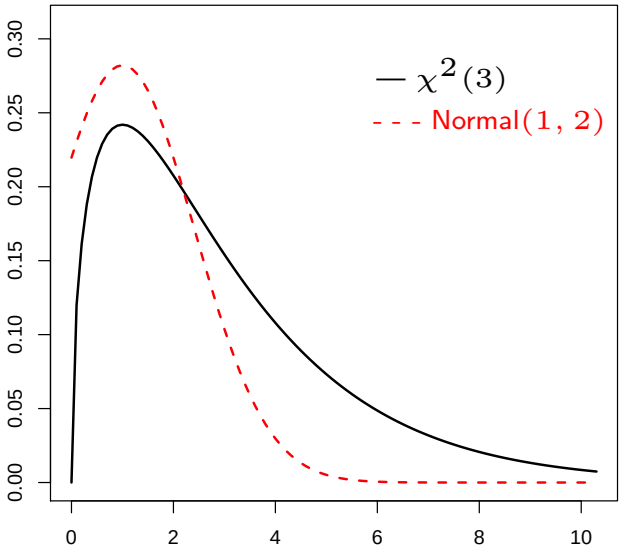

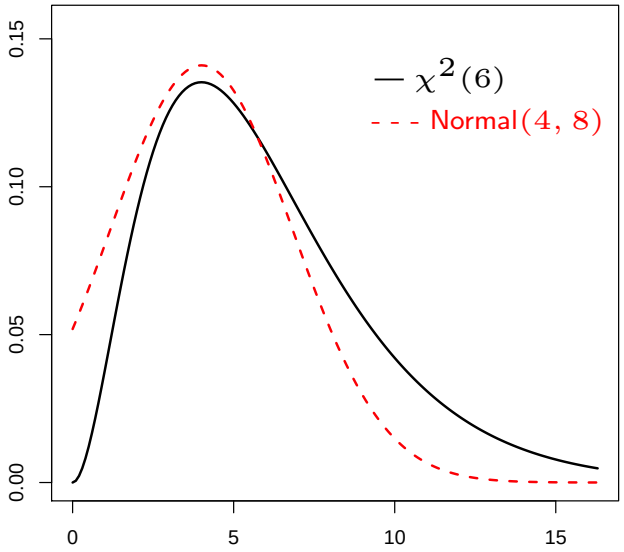

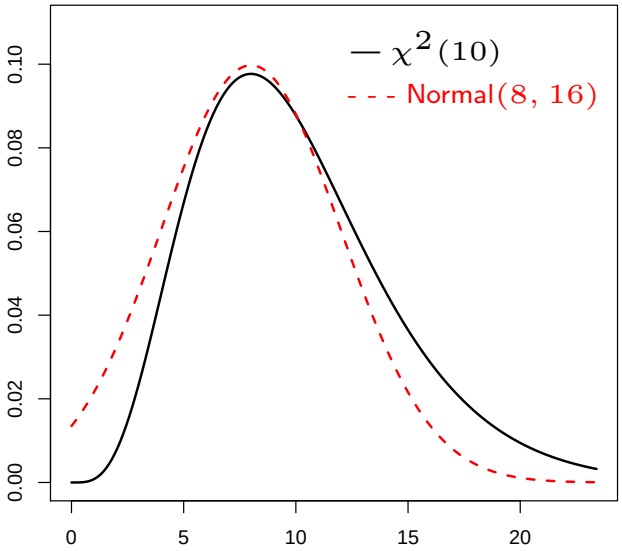

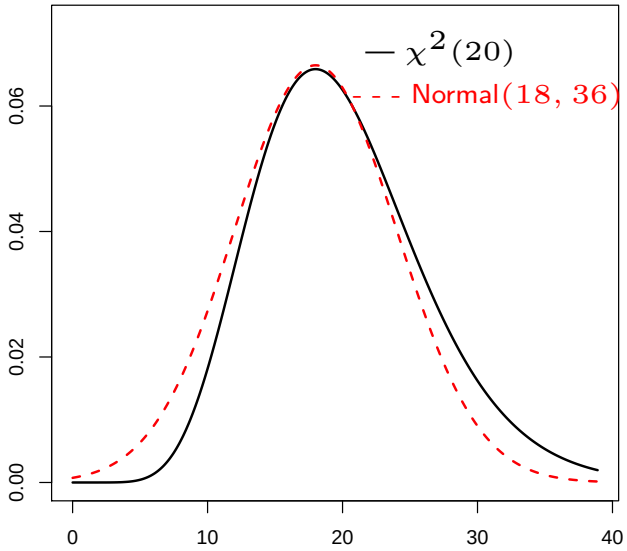

- The general idea is that using the fundamental probability equations, we can approximate a generic conditional (posterior) distribution as

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- This idea can be used to approximate any generic required posterior distribution

Objective of Bayesian estimation

- In a Bayesian LGM, the required distributions are

$$p(\theta_j \mid \boldsymbol{y}) = \int p(\theta_j, \boldsymbol{\psi} \mid \boldsymbol{y}) d\boldsymbol{\psi} = \int p(\boldsymbol{\psi} \mid \boldsymbol{y}) p(\theta_j \mid \boldsymbol{\psi}, \boldsymbol{y}) d\boldsymbol{\psi}$$

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(1.) can be easily estimated as

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where

- $\tilde{p}(\boldsymbol{\theta} \mid \boldsymbol{\psi}, \boldsymbol{y})$ is the Laplace approximation of $p(\boldsymbol{\theta} \mid \boldsymbol{\psi}, \boldsymbol{y})$
- $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(\boldsymbol{\psi})$ is its mode

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- This is the algorithm implemented by default by `R-INLA`, but this choice can be modified
  - If extra precision is required, it is possible to run the full Laplace approximation — of course at the expense of running time!

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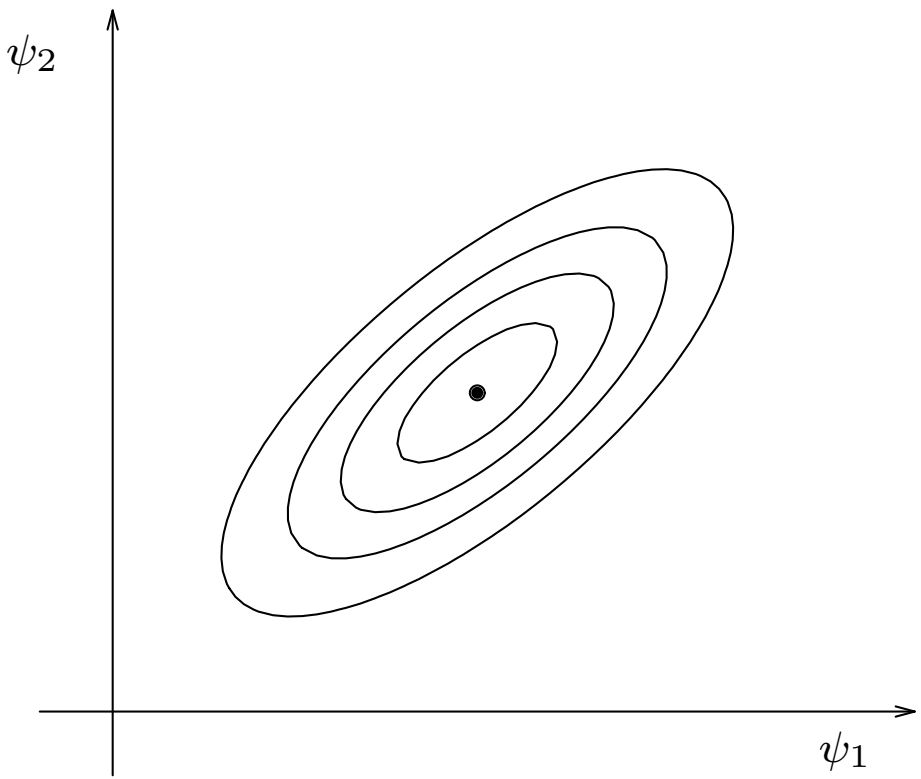

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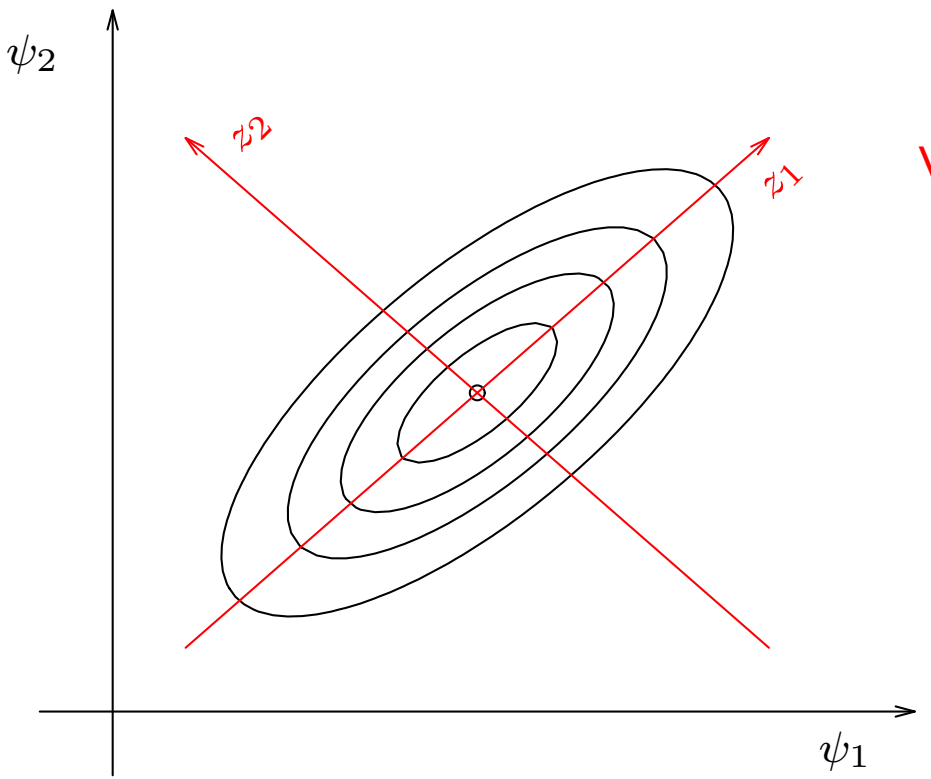

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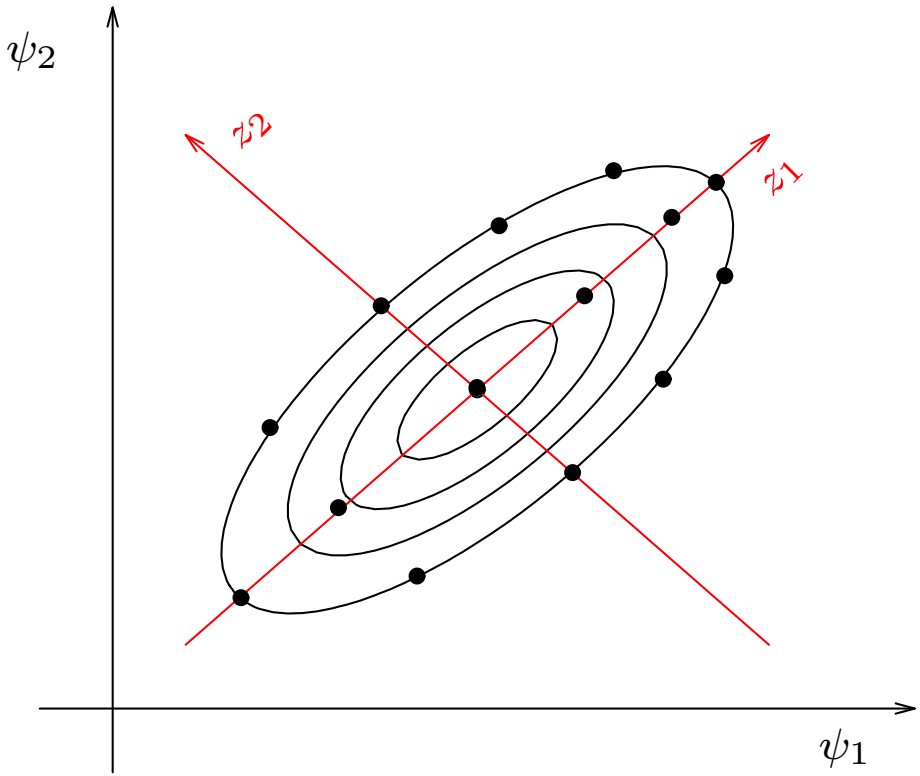

ii. For each element ψ_h^* in the grid,
   - Obtain the marginal posterior $\tilde{p}(\psi_h^* \mid \boldsymbol{y})$, using interpolation and possibly correcting for (probable) skewness by using log-splines;
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iii. Marginalise ψ_h^* to obtain the marginal posteriors $\tilde{p}(\theta_j \mid \boldsymbol{y})$ using **numerical integration**

$$\tilde{p}(\theta_j \mid \boldsymbol{y}) \approx \sum_{h=1}^{H} \tilde{p}(\theta_j \mid \psi_h^*, \boldsymbol{y}) \tilde{p}(\psi_h^* \mid \boldsymbol{y}) \Delta_h$$

Integrated Nested Laplace Approximation (INLA)

So, it's all in the name...

Integrated Nested Laplace Approximation

- Because Laplace approximation is the basis to estimate the unknown distributions
- Because the Laplace approximations are nested within one another
  - Since (2.) is needed to estimate (1.)
  - NB: Consequently the estimation of (1.) might not be good enough, but it can be refined
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- Suppose we want to make inference on a very simple model

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\begin{aligned}
y_{ij} \mid \theta_j, \psi &\sim \text{Normal}(\theta_j, \sigma_0^2) && (\sigma_0^2 \text{ assumed known}) \\
\theta_j \mid \psi &\sim \text{Normal}(0, \tau) && (\psi = \tau^{-1} \text{ is the precision}) \\
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- So, the model is made by a three-level hierarchy:
  1. Data $\boldsymbol{y} = (y_{ij})$ for $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$
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- **NB**: This model is in fact semi-conjugated, so inference is possible numerically or using simple MCMC algorithms

INLA — example

- Because of semi-conjugacy, we know that

$$\boldsymbol{\theta}, \boldsymbol{y} \mid \psi \sim \text{Normal}(\cdot, \cdot)$$

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$$\begin{aligned} p(\psi \mid \boldsymbol{y}) &\propto p(\boldsymbol{y} \mid \psi) p(\psi) \\ &\propto \frac{\overbrace{p(\boldsymbol{\theta}, \boldsymbol{y} \mid \psi)}^{\text{Gaussian}} p(\psi)}{\underbrace{p(\boldsymbol{\theta} \mid \boldsymbol{y}, \psi)}_{\text{Gaussian}}} \end{aligned}$$

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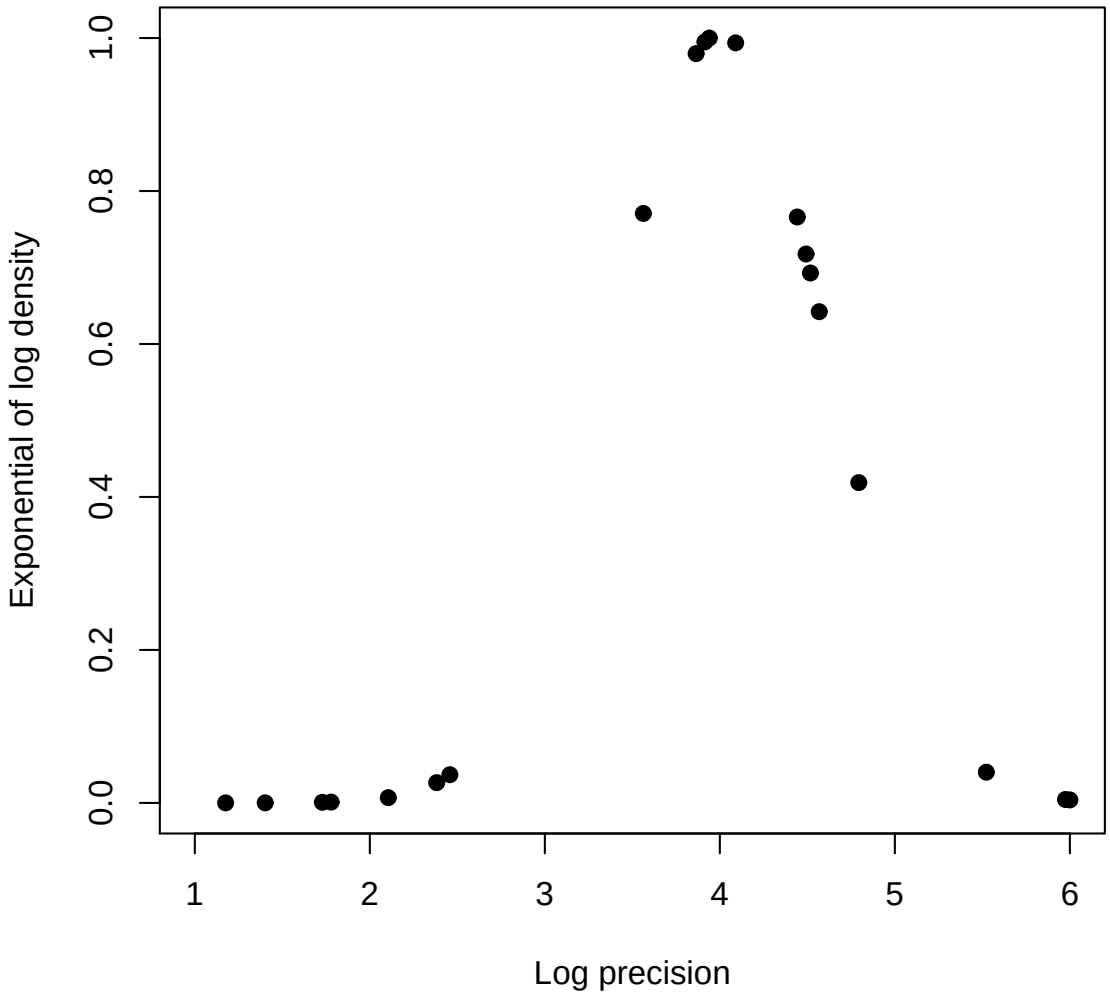
- Moreover, because $p(\boldsymbol{\theta} \mid \boldsymbol{y}) \sim \text{Normal}(\cdot, \cdot)$ and so are all the resulting marginals (ie for every element j), it is easy to compute

$$p(\theta_j \mid \boldsymbol{y}) = \int \underbrace{p(\theta_j \mid \boldsymbol{y}, \psi)}_{\text{Gaussian}} \underbrace{p(\psi \mid \boldsymbol{y})}_{\text{Approximated}} \, d\psi$$

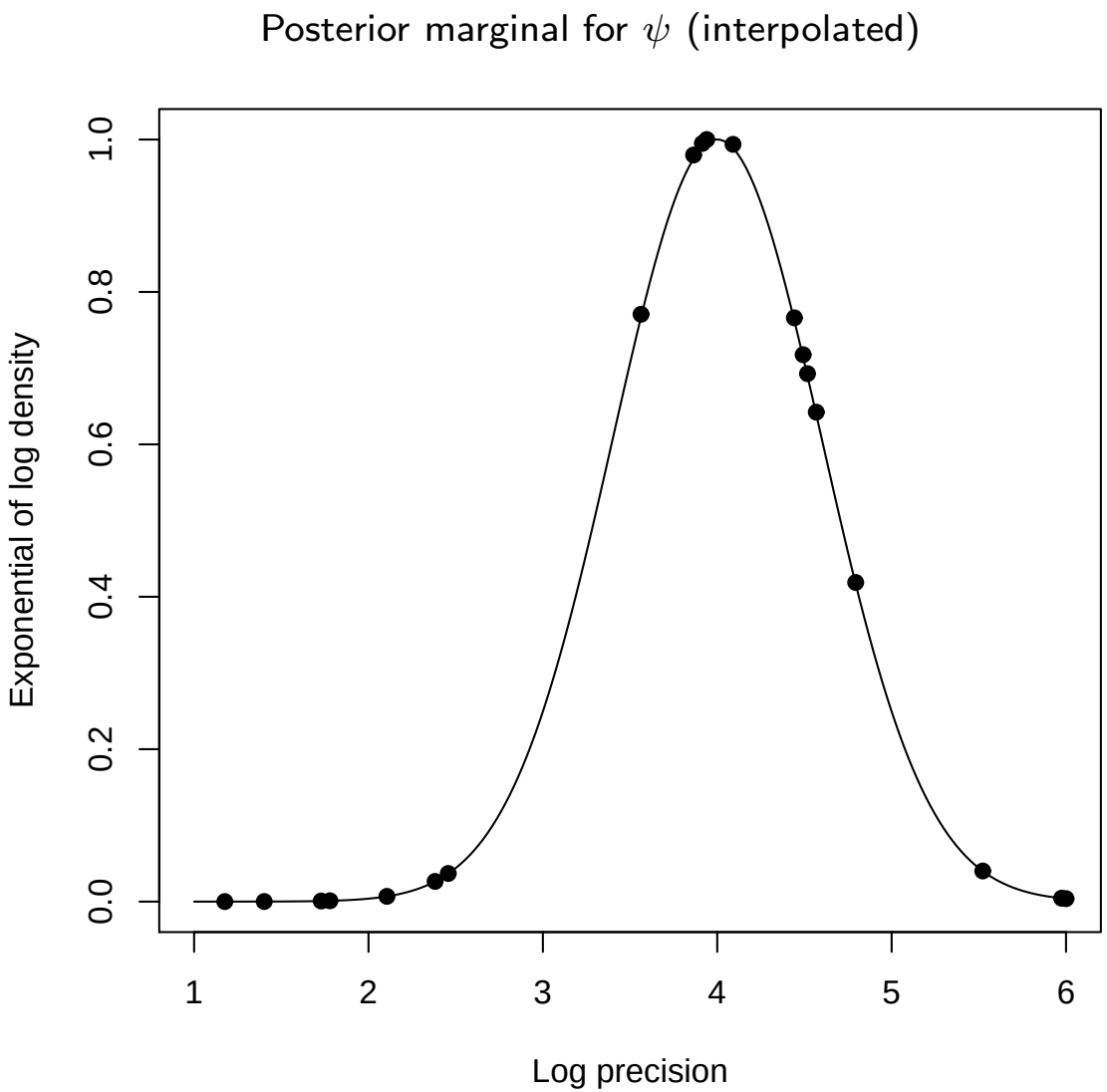
INLA — example

1. Select a grid of H points for ψ ($\{\psi_h^*\}$) and the associated area weights ($\{\Delta_h\}$)

Posterior marginal for ψ : $p(\psi \mid \boldsymbol{y}) \propto \frac{p(\boldsymbol{\theta}, \boldsymbol{y} \mid \psi) p(\psi)}{p(\boldsymbol{\theta} \mid \boldsymbol{y}, \psi)}$

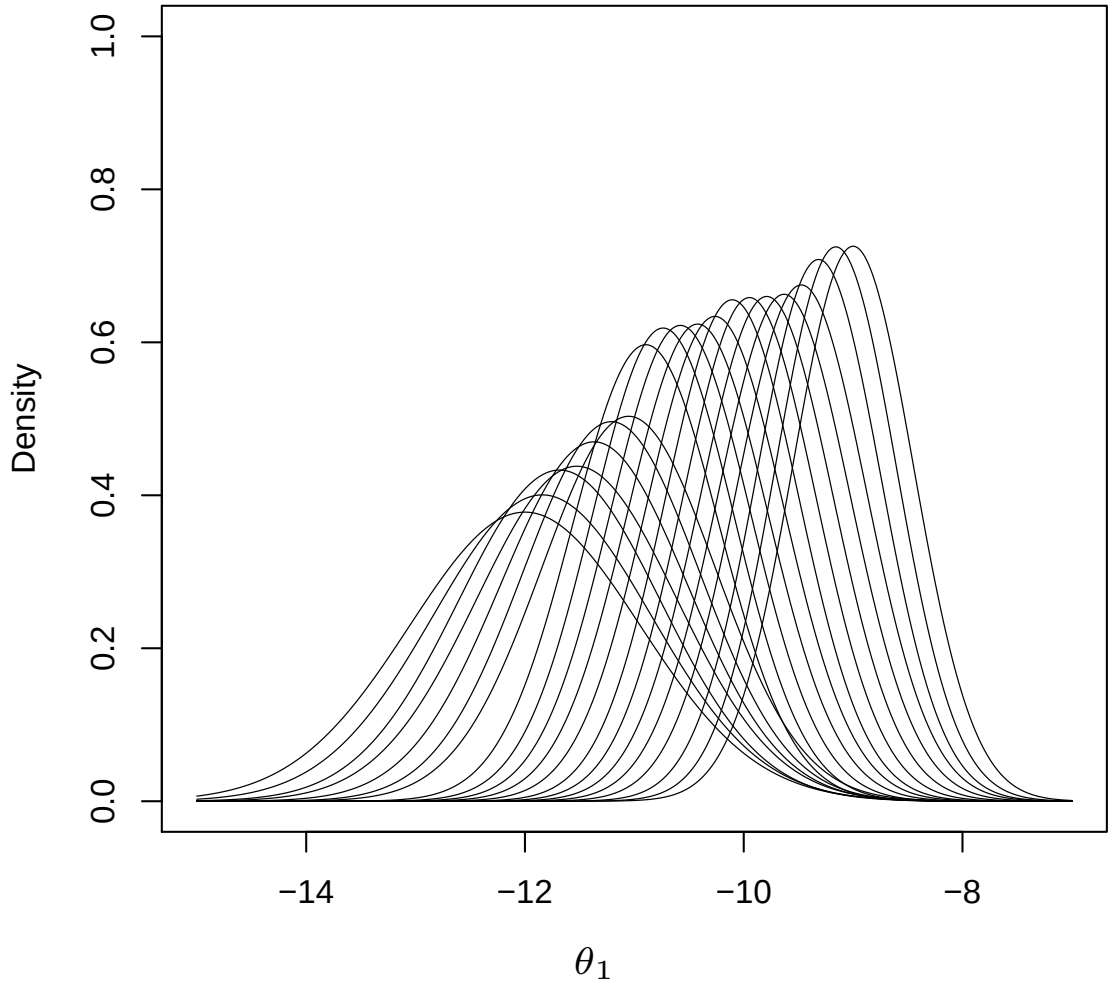


2. Interpolate the posterior density to compute the approximation to the posterior



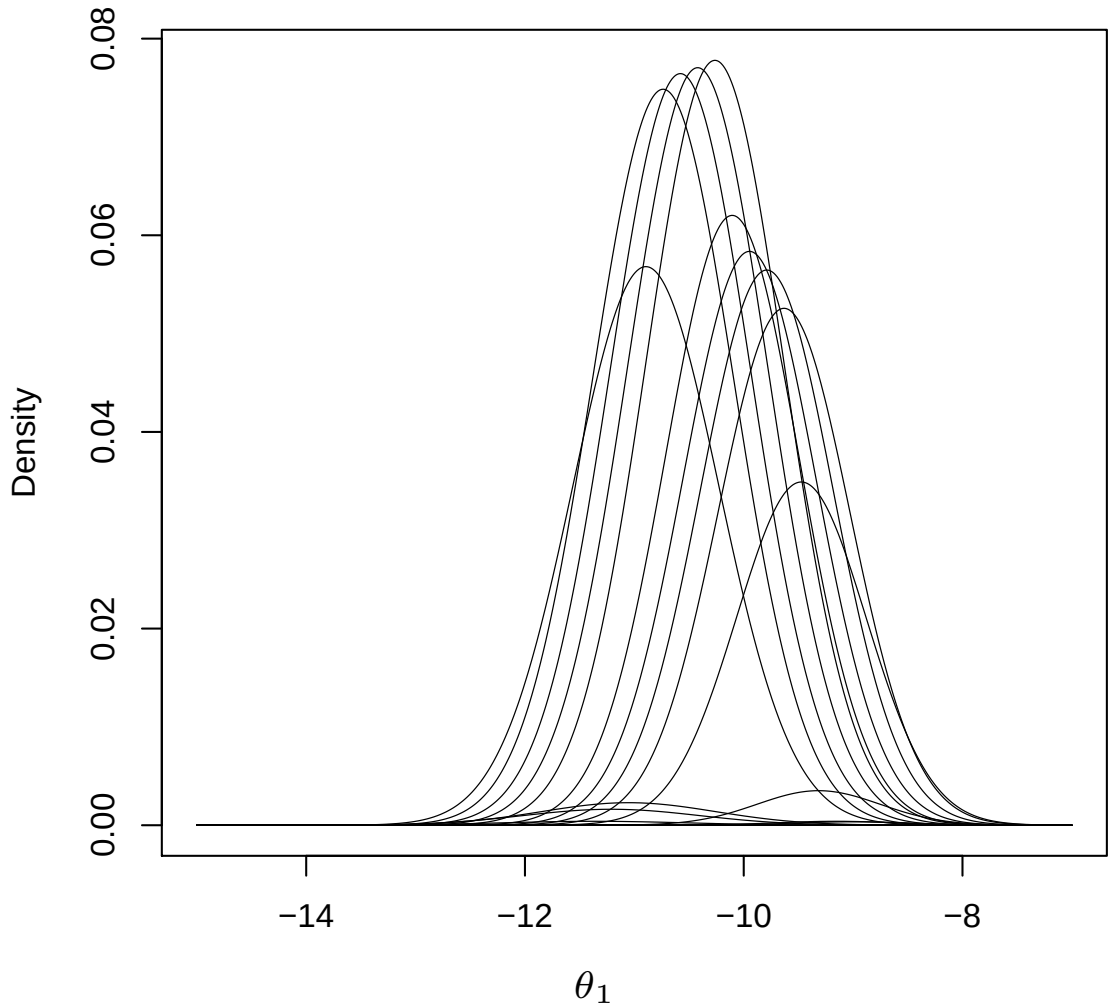
3. Compute the posterior marginal for each θ_j given each ψ on the $H-$dimensional grid

Posterior marginal for θ_1, conditional on each $\{\psi_h^*\}$ value (unweighted)



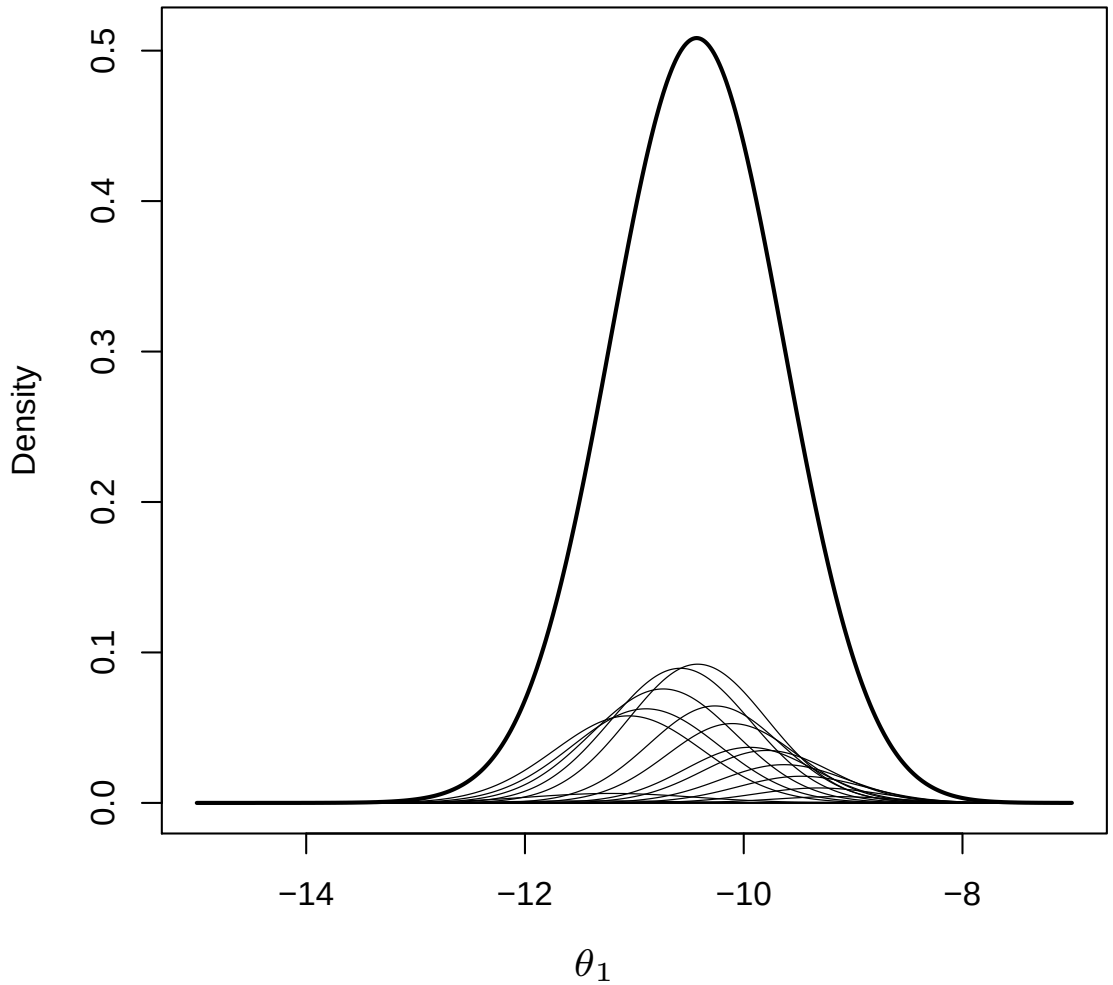
4. Weight the resulting (conditional) marginal posteriors by the density associated with each ψ on the grid

Posterior marginal for θ_1, conditional on each $\{\psi_h^*\}$ value (weighted)



5. (Numerically) sum over all the conditional densities to obtain the marginal posterior for each of the elements θ_j

Posterior marginal for θ_1 : $p(\theta_1 \mid \boldsymbol{y})$



INLA — Summary

- The basic idea behind the INLA procedure is simple
  - Repeatedly use Laplace approximation and take advantage of computational simplifications due to the structure of the model
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- Complications are mostly computational and occur when
  - Extending to more than one hyper-parameter
  - Markedly non-Gaussian observations

Using the package R-INLA

The INLA package for R

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1. The GMRFLib library
   - This is a C library for fast and exact simulation of GMRFs, used to perform
     - Unconditional simulation of a GMRF;
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   - A standalone C program that
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NB: Because the package R-INLA relies on a standalone C program, it is not available directly from CRAN

The INLA package for R — Installation

- Visit the website

  www.r-inla.org

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- Later, you can upgrade the package by typing
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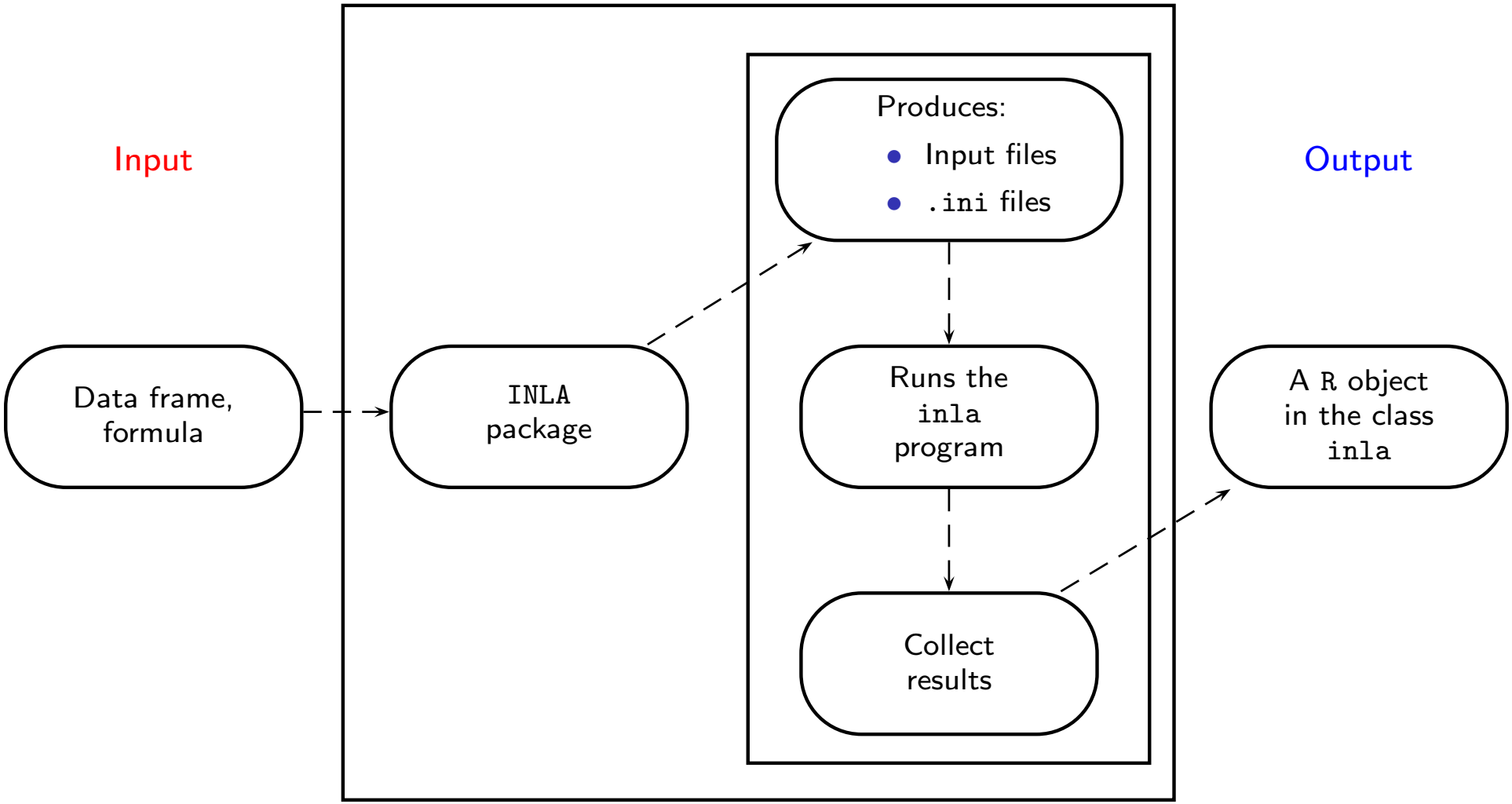
- R-INLA runs natively under Linux, Windows and Mac and it is possible to do multi-threading using OpenMP

The INLA package for R — How does it work?



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- Much of the recent development in R-INLA is devoted to extending the applications of INLA for spatial and spatio-temporal models as well as producing detailed information

- The website also has a discussion forum and a FAQ page

1. The first thing to do is to **specify the model**

- For example, assume we have a generic model

$$\begin{aligned} y_i &\overset{iid}{\sim} p(y_i \mid \theta_i) \\ \eta_i &= g(\theta_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + f(z_i) \end{aligned}$$

where

- $\boldsymbol{x} = (x_1, x_2)$ are observed covariates for which we are assuming a linear effect on some function $g(\cdot)$ of the parameter θ_i
- $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2) \sim \text{Normal}(0, \tau_1^{-1})$ are unstructured ("fixed") effects
- z is an **index**. This can be used to include structured ("random"), spatial, spatio-temporal effect, etc.
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- As mentioned earlier, this formulation can actually be used to represent quite a wide class of models!

Step by step guide to using R-INLA

- The model is translated in R code using a formula
- This is sort of standard in R (you would do pretty much the same for calls to functions such as lm, or glm, or lmer)

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formula = y ~ x1 + x2 + f(z, model=...)
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- The f() function can account for several structured effects
- This is done by specifying a different model
  - iid, iid1d, iid2d, iid3d specify random effects
  - rw1, rw2, ar1 are smooth effect of covariates or time effects
  - seasonal specifies a seasonal effect
  - besag models spatially structured effects (CAR)
  - generic is a user-defined precision matrix

Step by step guide to using R-INLA

2. Call the function `inla`, specifying the data and options (more on this later), eg

```
m = inla(formula, data=data.frame(y,x1,x2,z))
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```

- The data need to be included in a suitable `data.frame`
- `R` returns an object `m` in the class `inla`, which has some methods available
  - `summary()`
  - `plot()`
- The options let you specify the priors and hyperpriors, together with additional output

Step by step guide to using R-INLA

```
names(m)
 [1] "names.fixed"                  "summary.fixed"
 [3] "marginals.fixed"              "summary.lincomb"
 [5] "marginals.lincomb"            "size.lincomb"
 [7] "summary.lincomb.derived"      "marginals.lincomb.derived"
 [9] "size.lincomb.derived"         "mlik"
[11] "cpo"                          "model.random"
[13] "summary.random"               "marginals.random"
[15] "size.random"                  "summary.linear.predictor"
[17] "marginals.linear.predictor"   "summary.fitted.values"
[19] "marginals.fitted.values"      "size.linear.predictor"
[21] "summary.hyperpar"             "marginals.hyperpar"
[23] "internal.summary.hyperpar"    "internal.marginals.hyperpar"
[25] "si"                           "offset.linear.predictor"
[27] "model.spde2.blc"              "summary.spde2.blc"
[29] "marginals.spde2.blc"          "size.spde2.blc"
[31] "logfile"                      "misc"
[33] "dic"                          "mode"
[35] "neffp"                        "joint.hyper"
[37] "nhyper"                       "version"
[39] "Q"                            "graph"
[41] "cpu.used"                     ".args"
[43] "call"                         "model.matrix"
```

First, generate some data from an assumed model

$$y_i \sim \text{Binomial}(\pi_i, N_i), \qquad \text{for } i = 1, \ldots, n = 12$$

```
library(INLA)

# Data generation
n=12
Ntrials = sample(c(80:100), size=n, replace=TRUE)
eta = rnorm(n,0,0.5)
prob = exp(eta)/(1 + exp(eta))
y = rbinom(n, size=Ntrials, prob = prob)
data=data.frame(y=y,z=1:n,Ntrials)
```

Example — Binary data with individual random effect

```
data
```

	y	z	Ntrials
1	50	1	95
2	37	2	97
3	36	3	93
4	47	4	96
5	39	5	80
6	67	6	97
7	60	7	89
8	57	8	84
9	34	9	89
10	57	10	96
11	46	11	87
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We want to fit the following model

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y_i &\sim \text{Binomial}(\pi_i, N_i), && \text{for } i = 1, \ldots, n = 12 \\
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p(\sigma^2) &\propto \sigma^{-2} = \tau && (\text{“non-informative” prior}) \\
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```
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m=inla(formula, data=data,
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       control.predictor = list(compute = TRUE))
summary(m)
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Example — Binary data with individual random effect

```
Call:
c("inla(formula = formula, family = \"binomial\", data = data, Ntrials = Ntrials,
  "control.predictor = list(compute = TRUE))")

Time used:
 Pre-processing    Running inla Post-processing           Total
         0.2258          0.0263          0.0744          0.3264

Fixed effects:
               mean    sd 0.025quant 0.5quant 0.975quant kld
(Intercept) -0.0021 0.136     -0.272  -0.0021      0.268   0

Random effects:
Name   Model
 z   IID model

Model hyperparameters:
                mean   sd     0.025quant 0.5quant 0.975quant
Precision for z  7.130  4.087  2.168      6.186    17.599

Expected number of effective parameters(std dev): 9.494(0.7925)
Number of equivalent replicates : 1.264

Marginal Likelihood:  -54.28
CPO and PIT are computed

Posterior marginals for linear predictor and fitted values computed
```

Exploring the R-INLA output

```
Fixed effects:
               mean    sd 0.025quant 0.5quant 0.975quant kld
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Exploring the R-INLA output

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Fixed effects:
              mean    sd 0.025quant 0.5quant 0.975quant kld
(Intercept) -0.0021 0.136     -0.272  -0.0021      0.268   0
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- For each unstructured ("fixed") effect, R-INLA reports a set of summary statistics from the posterior distribution

Exploring the R-INLA output

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Fixed effects:
               mean    sd 0.025quant 0.5quant 0.975quant kld
(Intercept) -0.0021 0.136     -0.272  -0.0021      0.268   0
```

- For each unstructured ("fixed") effect, R-INLA reports a set of summary statistics from the posterior distribution
- The value of the Kullback-Leibler divergence (KLD) describes the difference between the standard Gaussian and the Simplified Laplace Approximation to the marginal posterior densities
  - Small values indicate that the posterior distribution is well approximated by a Normal distribution
  - If so, the more sophisticated SLA gives a "good" error rate and therefore there is no need to use the more computationally intensive "full" Laplace approximation

Exploring the R-INLA output

```
Random effects:
Name   Model
 z   IID model

Model hyperparameters:
                  mean    sd     0.025quant 0.5quant 0.975quant
Precision for z  7.130  4.087  2.168       6.186    17.599
```

- Also for each hyper-parameter, the summary statistics are reported to describe the posterior distribution
- **NB**: INLA reports results on the **precision** scale (more on this later)

Exploring the R-INLA output

```
Expected number of effective parameters(std dev): 9.494(0.7925)
Number of equivalent replicates : 1.264
```

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- The expected number of effective parameters is basically the number of **independent** parameters included in the model
  - In a hierarchical model, because of shrinkage, information is shared across parameters
  - Example: in this case there are 14 actual parameters ($\alpha, \sigma^2, f(1), \ldots, f(12)$). However, because the structured effects are exchangeable (ie correlated) the "effective" number of parameters is (on average) just 9.5

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- The number of equivalent replicates indicates the available information (in terms of sample size) per effective parameter
  - Example: there are 12 data points and on average 9.5 parameters; so each is estimated using on average $12/9.5 \approx 1.3$ data points
  - Low values (with respect to the overall sample size) are indicative of poor fit

Exploring the R-INLA output

```
Marginal Likelihood:  -54.28
CPO and PIT are computed
```

- R-INLA can produce two types of "leave-one-out" measures of fit
  1. Conditional Predictive Ordinate (CPO): $p(y_i \mid \boldsymbol{y}_{-i})$
     - "Extreme" values for CPO indicate a surprising observation
  2. Probability Integral Transforms (PIT): $\Pr(y_i^{\text{new}} \leq y_i \mid \boldsymbol{y}_{-i})$
     - "Extreme" values for PIT indicate outliers
     - A histogram of PIT that does not look Uniformly distributed indicate lack of fit for the current model

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     - "Extreme" values for PIT indicate outliers
     - A histogram of PIT that does not look Uniformly distributed indicate lack of fit for the current model

- If the option

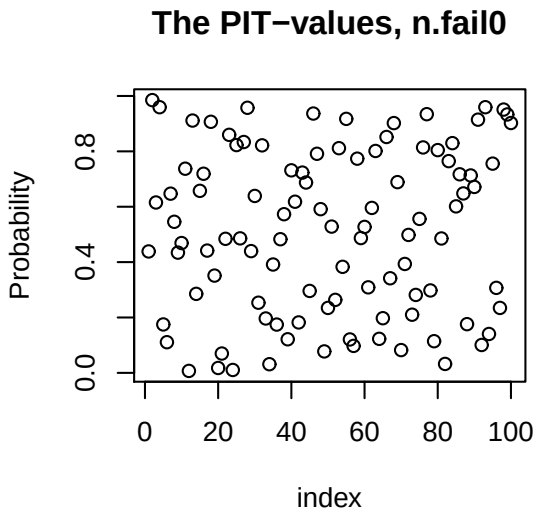
  `control.compute=list(cpo=TRUE)`

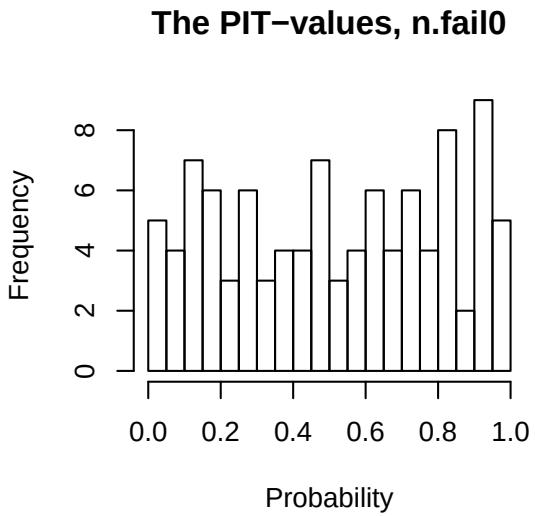
  is added to the call to the function `inla` then the resulting object contains values for CPO and PIT, which can then be post-processed
  - **NB**: for the sake of model checking, it is useful to to increase the accuracy of the estimation for the tails of the marginal distributions
  - This can be done by adding the option

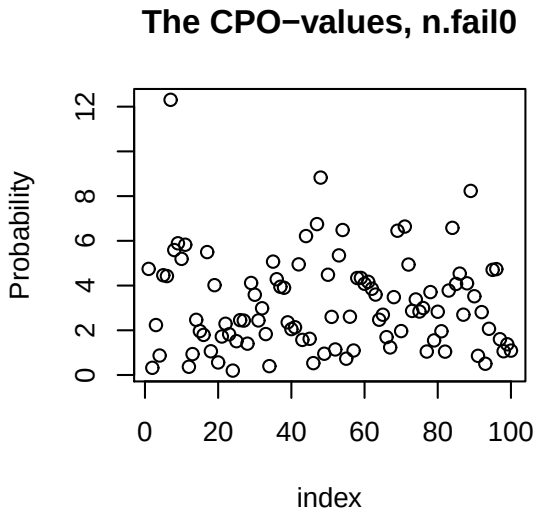
    `control.inla = list(strategy = "laplace", npoints = 21)`

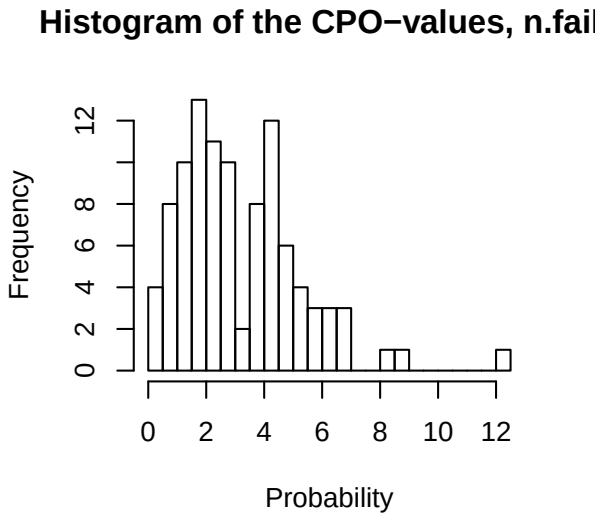
    to add more evaluation points (npoints=21) instead of the default npoints=9

Exploring the R-INLA output





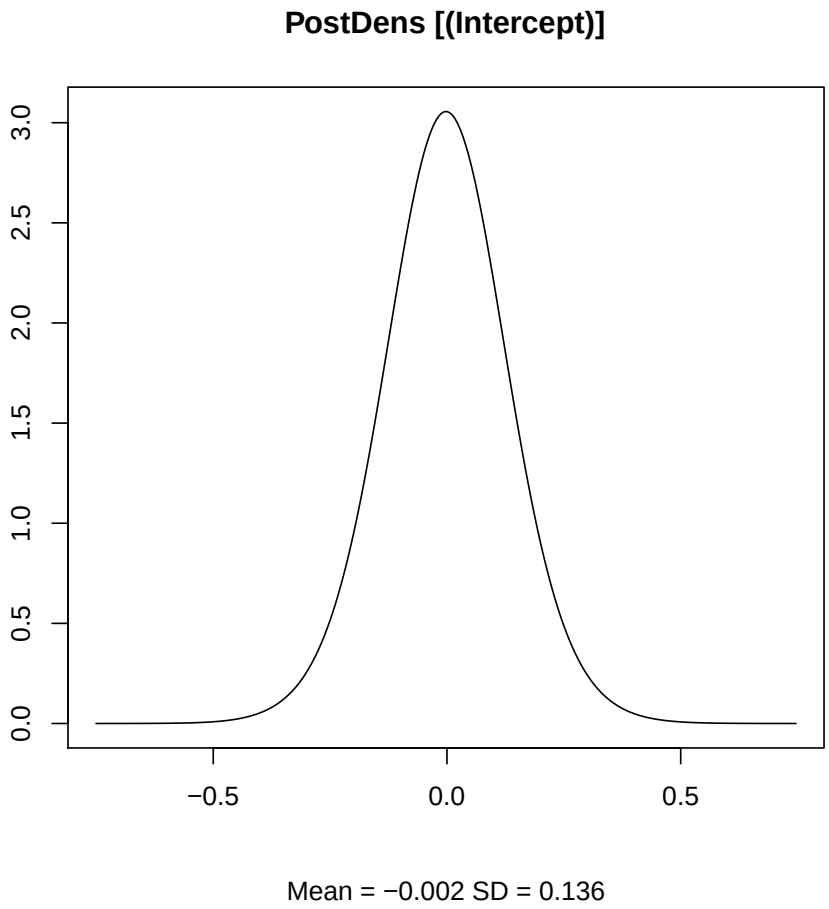




```
plot(m)

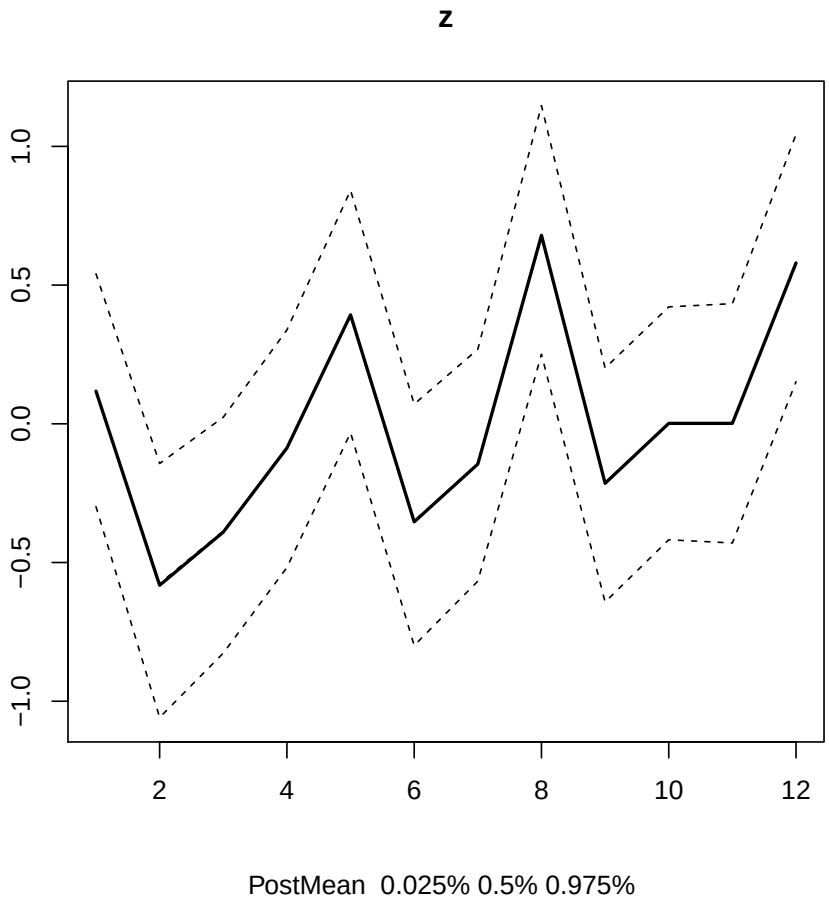

plot(m,
     plot.fixed.effects = TRUE,
     plot.lincomb = FALSE,
     plot.random.effects = FALSE,
     plot.hyperparameters = FALSE,
     plot.predictor = FALSE,
     plot.q = FALSE,
     plot.cpo = FALSE
)


plot(m,single = TRUE)
```



```
plot(m)
```

```
plot(m,
     plot.fixed.effects = FALSE,
     plot.lincomb = FALSE,
     plot.random.effects = TRUE,
     plot.hyperparameters = FALSE,
     plot.predictor = FALSE,
     plot.q = FALSE,
     plot.cpo = FALSE
)
```



Manipulating the results from R-INLA

- The elements of the object `m` can be used for post-processing

```
m$summary.fixed
```

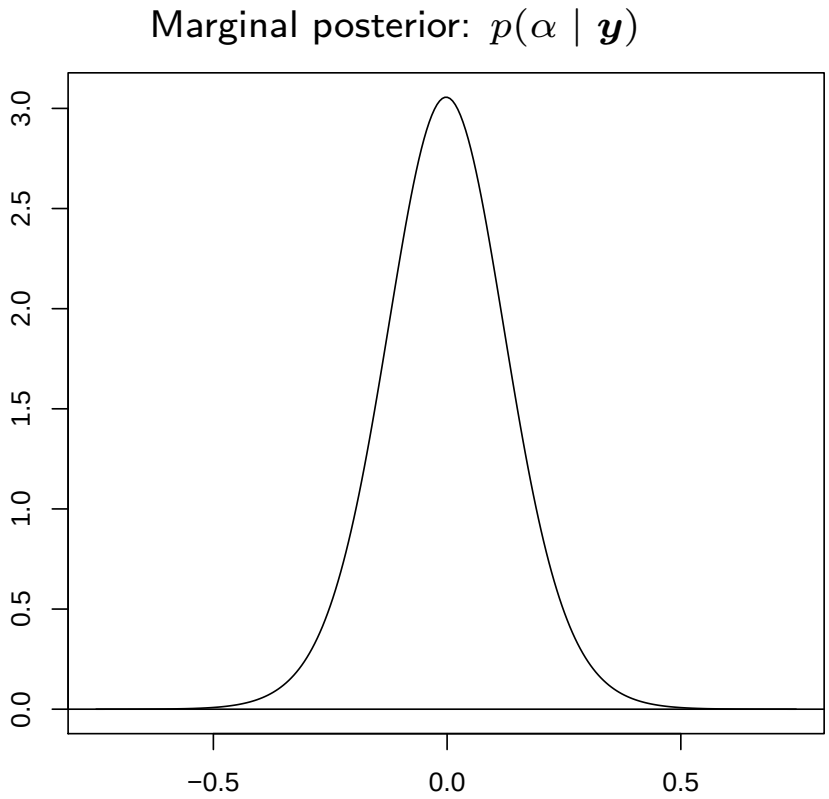
```
                     mean        sd 0.025quant     0.5quant 0.975quant          kld
(Intercept) -0.002092578 0.1360447 -0.2720331 -0.002101465  0.2680023 1.866805e-08
```

```
m$summary.random

$z
   ID         mean        sd  0.025quant     0.5quant  0.975quant          kld
1   1  0.117716597 0.2130482 -0.29854459  0.116540837  0.54071007 1.561929e-06
2   2 -0.582142549 0.2328381 -1.05855344 -0.575397613 -0.14298960 3.040586e-05
3   3 -0.390419424 0.2159667 -0.82665552 -0.386498698  0.02359256 1.517773e-05
4   4 -0.087199172 0.2174477 -0.51798771 -0.086259111  0.33838724 7.076793e-07
5   5  0.392724605 0.2220260 -0.03217954  0.388462164  0.84160800 1.604348e-05
6   6 -0.353323459 0.2210244 -0.79933142 -0.349483252  0.07088015 1.242953e-05
7   7 -0.145238917 0.2122322 -0.56726042 -0.143798605  0.26859415 2.047815e-06
8   8  0.679294456 0.2279863  0.25076022  0.672226639  1.14699903 4.145645e-05
9   9 -0.214441626 0.2141299 -0.64230245 -0.212274011  0.20094086 4.577080e-06
10 10  0.001634115 0.2131451 -0.41797579  0.001622300  0.42152562 4.356243e-09
11 11  0.001593724 0.2190372 -0.42961274  0.001581019  0.43309253 3.843622e-09
12 12  0.580008923 0.2267330  0.15173745  0.573769187  1.04330359 3.191737e-05
```

Manipulating the results from R-INLA

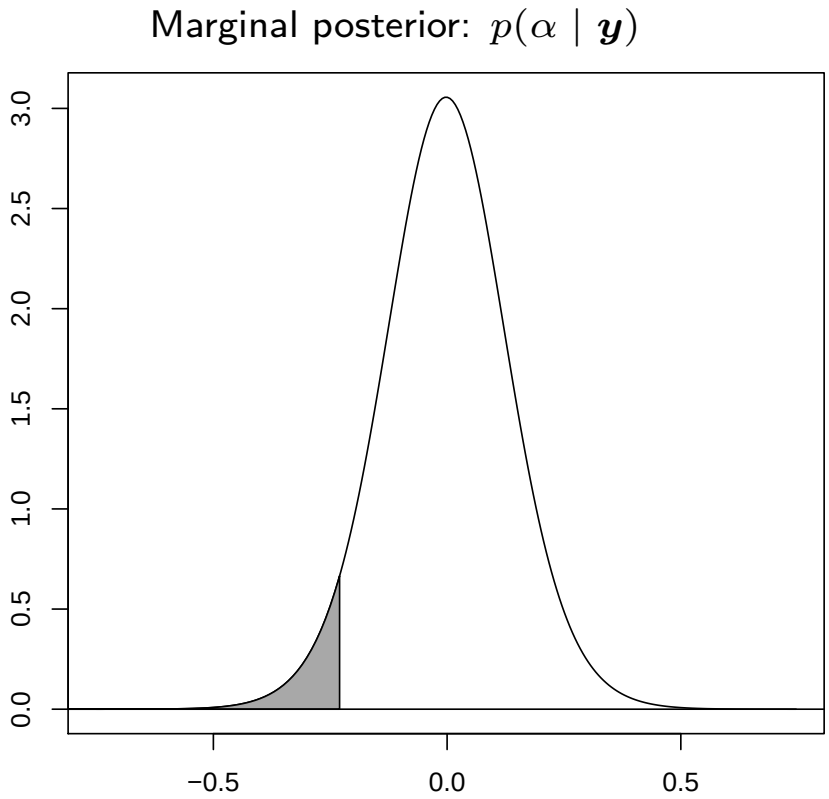
```
alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha),t="l")
```



Manipulating the results from R-INLA

```
alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha),t="l")
```

```
inla.qmarginal(0.05,alpha)
[1] -0.2257259
```

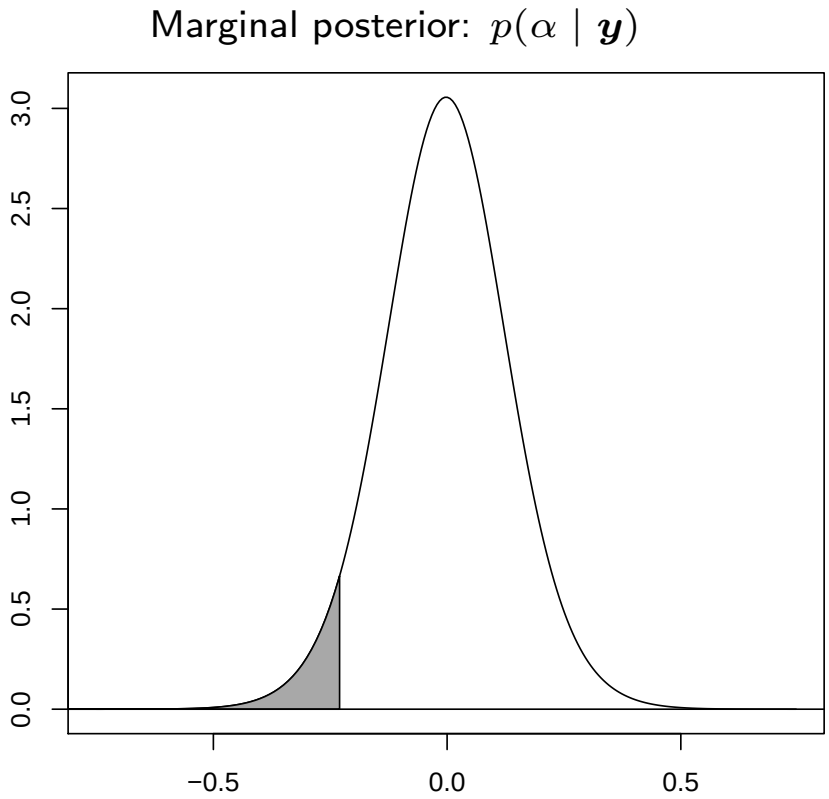


Manipulating the results from R-INLA

```
alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha),t="l")
```

```
inla.qmarginal(0.05,alpha)
[1] -0.2257259
```

```
inla.pmarginal(-.2257259,alpha)
[1] 0.04999996
```



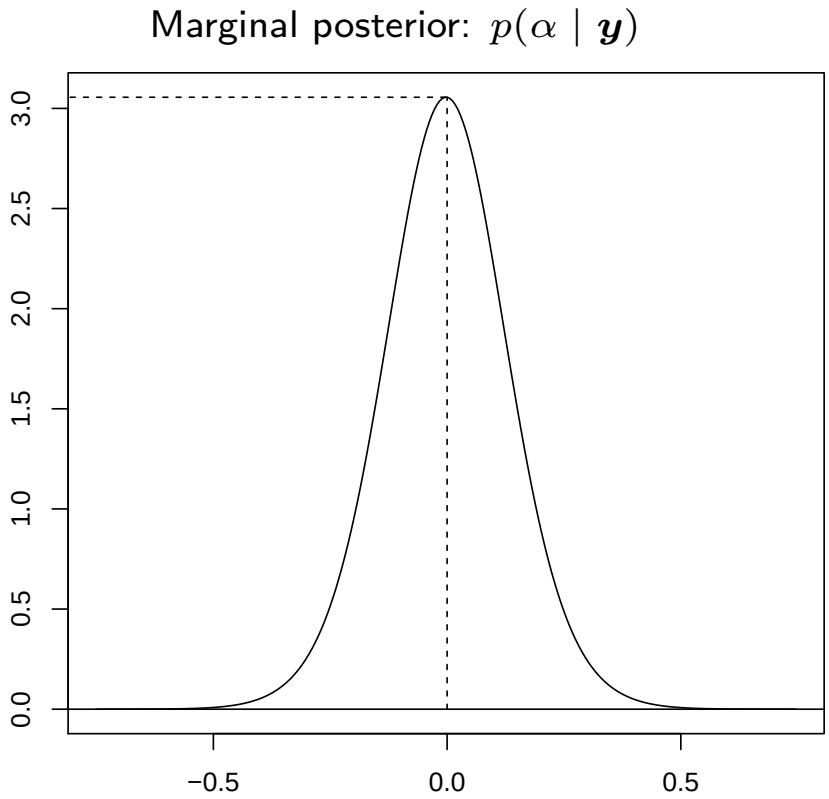
Manipulating the results from R-INLA

```
alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha),t="l")
```

```
inla.qmarginal(0.05,alpha)
[1] -0.2257259
```

```
inla.pmarginal(-.2257259,alpha)
[1] 0.04999996
```

```
inla.dmarginal(0,alpha)
[1] 3.055793
```



Manipulating the results from R-INLA

```
alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha),t="l")
```

```
inla.qmarginal(0.05,alpha)
[1] -0.2257259
```

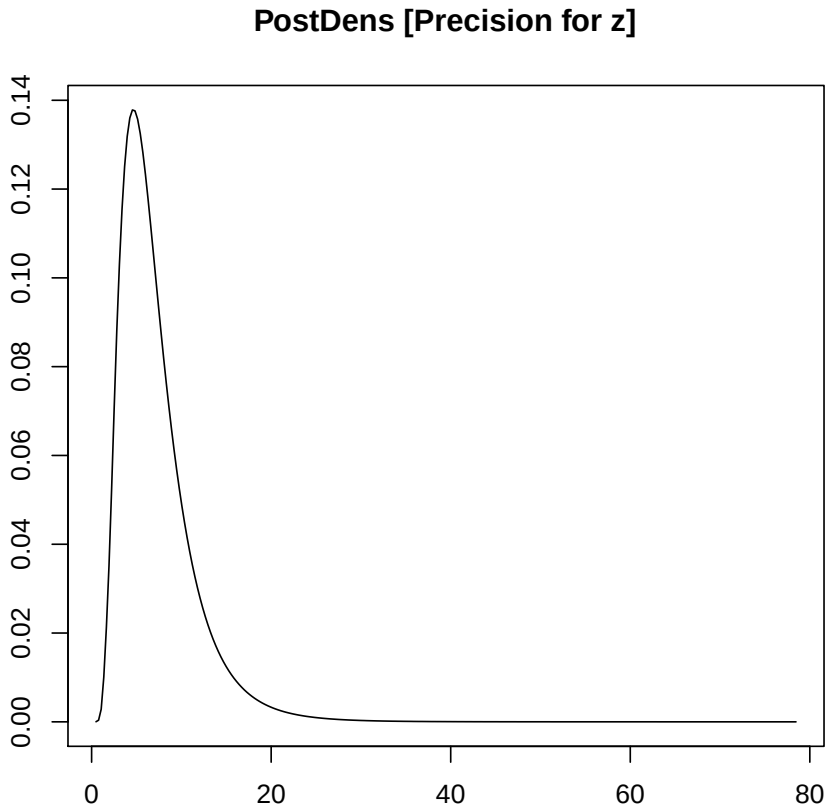
```
inla.pmarginal(-.2257259,alpha)
[1] 0.04999996
```

```
inla.dmarginal(0,alpha)
[1] 3.055793
```

```
inla.rmarginal(4,alpha)
[1]  0.05307452  0.07866796  -0.09931744 -0.02027463
```

NB: INLA works by default with **precisions**

```
plot(m,
     plot.fixed.effects = FALSE,
     plot.lincomb = FALSE,
     plot.random.effects = FALSE,
     plot.hyperparameters = TRUE,
     plot.predictor = FALSE,
     plot.q = FALSE,
     plot.cpo = FALSE
)
```



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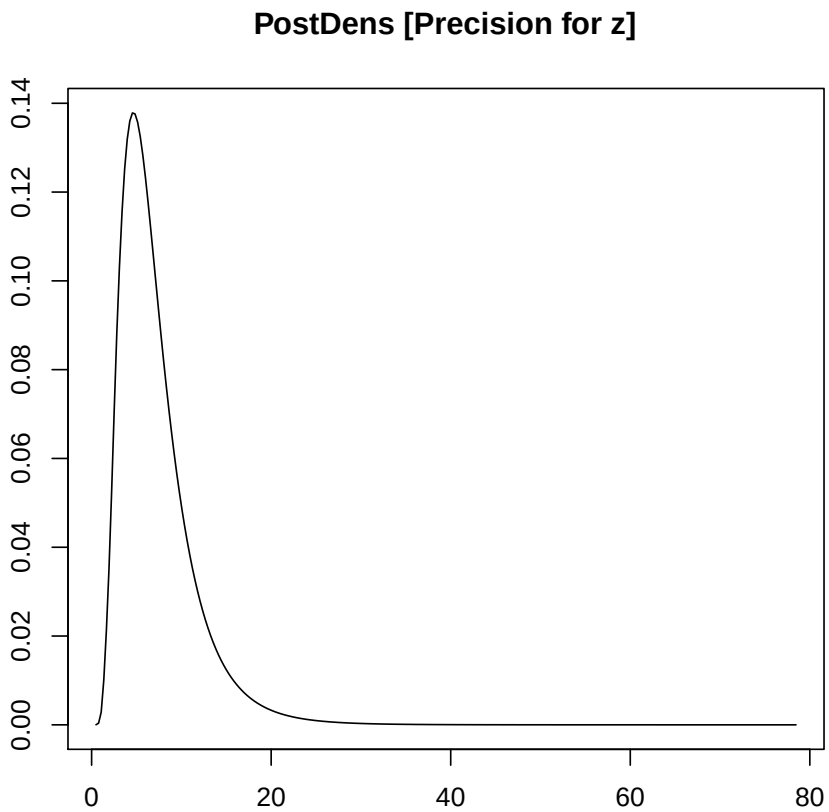

Problem: usually, we want to make inference on more interpretable parameters, eg **standard deviations**

- Using some built-in INLA functions
  - model$marginals.hyperpar
  - inla.expectation
  - inla.rmarginal

  it is possible to compute the structured variability, for example on the standard deviation scale, based on nsamples (default=1000) MC simulations from the estimated precision

```
s <- inla.contrib.sd(m,nsamples=1000)
s$hyper
```

```
                mean        sd      2.5%     97.5%
sd for z 0.416862 0.1098968 0.2332496 0.6478648
```

- Using some built-in INLA functions
  - model$marginals.hyperpar
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s$hyper
```
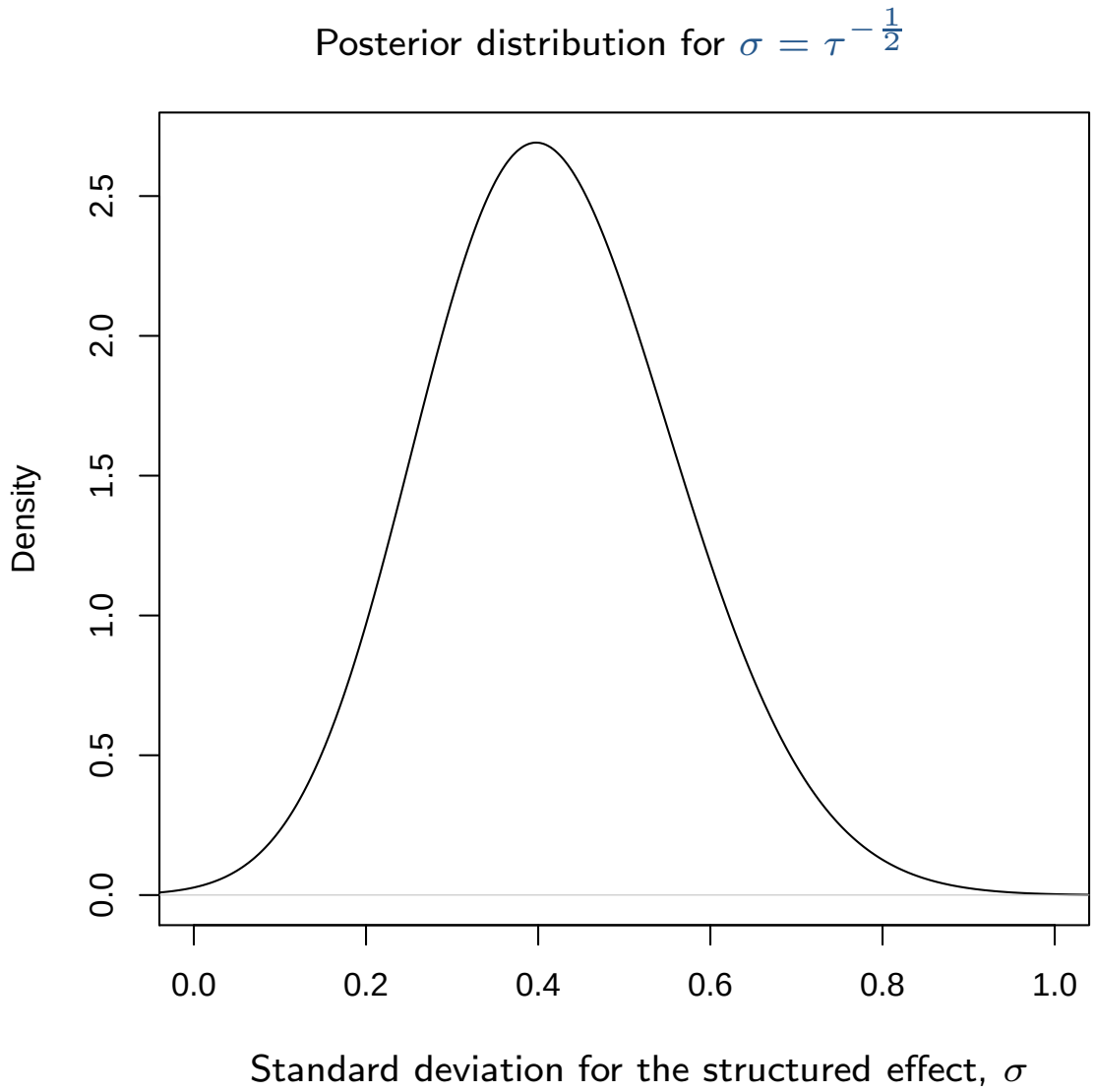

```
               mean        sd      2.5%     97.5%
sd for z 0.416862 0.1098968 0.2332496 0.6478648
```

- The object s contains a vector of simulations from the induced posterior distribution for the standard deviation scale, than can then be used for plots

```
hist(s$samples)
plot(density(s$samples,bw=.1),xlab="sigma",main="")
```


Posterior distribution for $\sigma = \tau^{-\frac{1}{2}}$
Density
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0
Standard deviation for the structured effect, σ

Example — Binary data with individual random effect

If we wanted to perform MCMC on this model, we could

1. Program it in JAGS/BUGS and save it as `model.txt`

```
model {
    for (i in 1:n) {
        y[i] ~ dbinom(pi[i],Ntrials[i])
        logit(pi[i]) <- alpha+f[i]
        f[i] ~ dnorm(0,tau)
    }
    alpha ~ dnorm(0,.001)
    log.sigma ~ dunif(0,10000)
    sigma <- exp(log.sigma)
    tau <- pow(sigma,-2)
}
```

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    }
    alpha ~ dnorm(0,.001)
    log.sigma ~ dunif(0,10000)
    sigma <- exp(log.sigma)
    tau <- pow(sigma,-2)
}
```

2. In R, use the library R2jags (or R2WinBUGS) to interface with the MCMC software

```
library(R2jags)
filein <- "model.txt"
dataJags <- list(y=y,n=n,Ntrials=Ntrials,prec=prec)
params <- c("sigma","tau","f","pi","alpha")
inits <- function(){ list(log.sigma=runif(1),alpha=rnorm(1),f=rnorm(n,0,1)) }
n.iter <- 100000
n.burnin <- 9500
n.thin <- floor((n.iter-n.burnin)/500)
mj <- jags(dataJags, inits, params, model.file=filein,n.chains=2, n.iter, n.burnin,
    n.thin, DIC=TRUE, working.directory=working.dir, progress.bar="text")
print(mj,digits=3,intervals=c(0.025, 0.975))
```

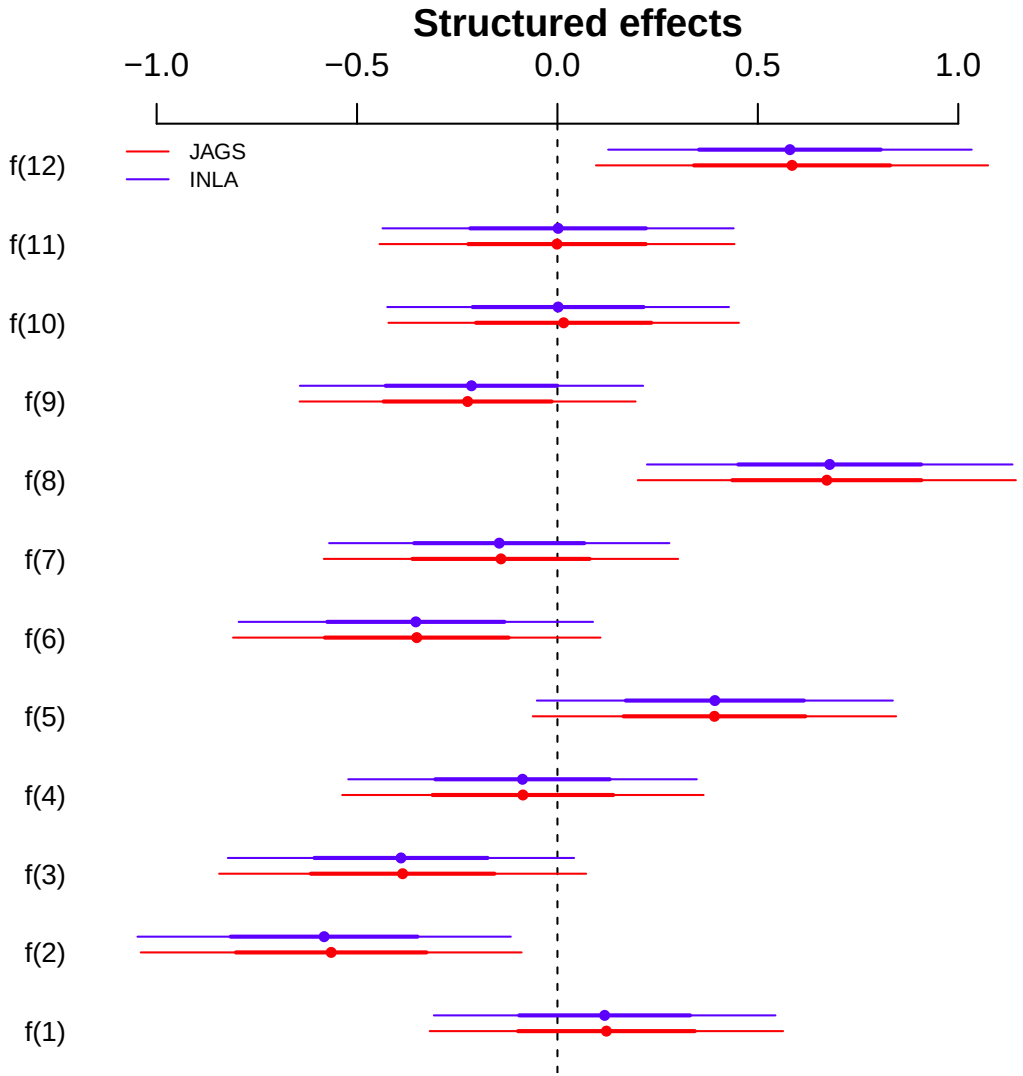
Example — Binary data with individual random effect

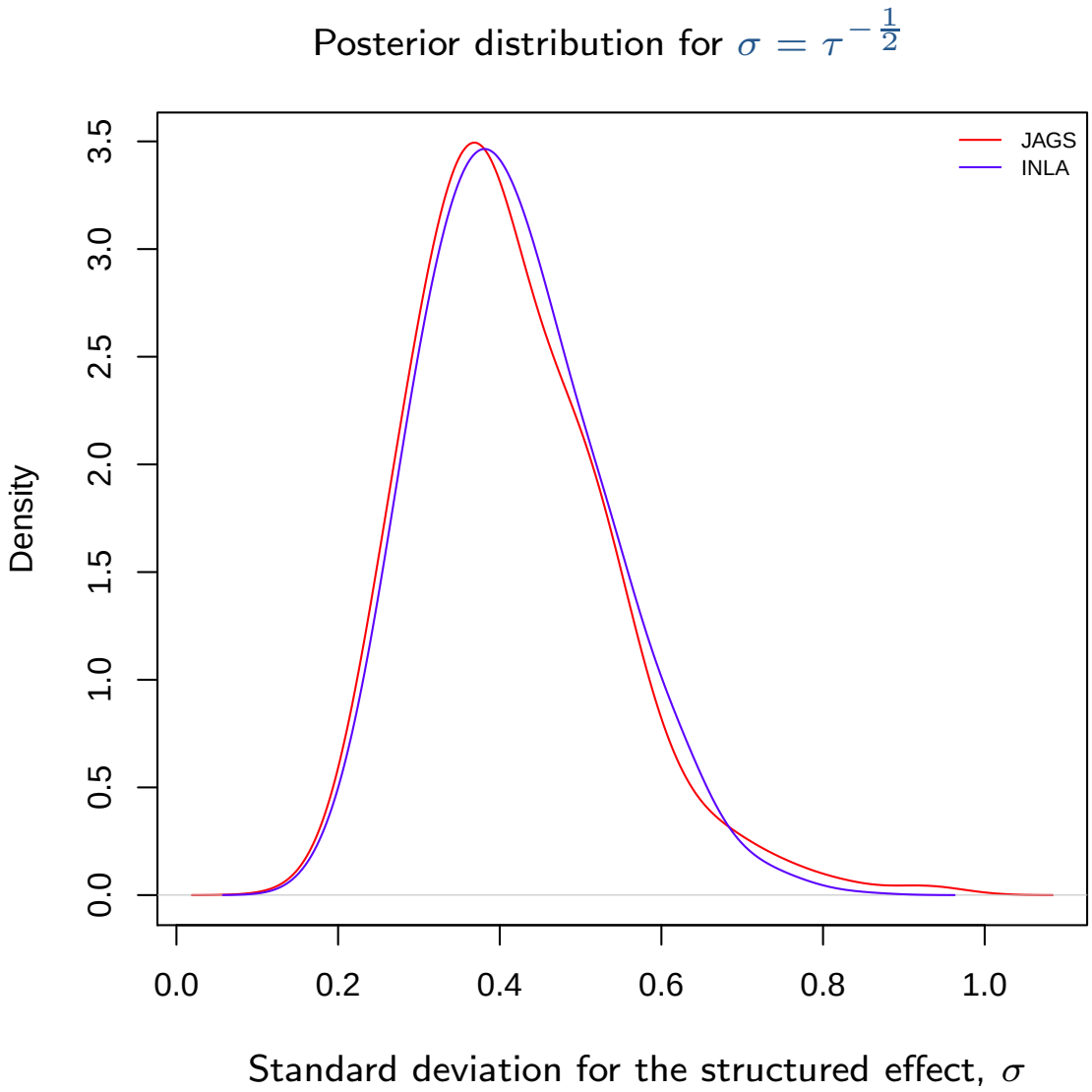
```
Inference for Bugs model at "model.txt", fit using jags,
 2 chains, each with 1e+05 iterations (first 9500 discarded), n.thin = 181
 n.sims = 1000 iterations saved                    (Time to run: 4.918 sec)

          mu.vect sd.vect   2.5%  97.5%  Rhat n.eff
alpha      -0.005   0.146 -0.270  0.292 1.001  1000
f[1]        0.122   0.220 -0.347  0.582 1.001  1000
f[2]       -0.564   0.238 -1.051 -0.115 1.008   190
f[3]       -0.386   0.229 -0.880  0.050 1.000  1000
f[4]       -0.086   0.225 -0.549  0.367 1.002   780
f[5]        0.392   0.227 -0.047  0.828 1.002   870
f[6]       -0.351   0.229 -0.805  0.081 1.000  1000
f[7]       -0.141   0.221 -0.578  0.286 1.001  1000
f[8]        0.672   0.236  0.246  1.200 1.002   860
f[9]       -0.224   0.210 -0.643  0.178 1.000  1000
f[10]       0.016   0.219 -0.396  0.463 1.006  1000
f[11]      -0.001   0.221 -0.441  0.416 1.002   780
f[12]       0.585   0.245  0.153  1.093 1.001  1000
sigma       0.414   0.120  0.230  0.693 1.000  1000
tau         7.415   4.546  2.080 18.951 1.000  1000
deviance   72.378   5.497 64.016 84.715 1.000  1000

For each parameter, n.eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor (at convergence, Rhat=1).

DIC info (using the rule, pD = var(deviance)/2)
pD = 15.1 and DIC = 87.5
DIC is an estimate of expected predictive error (lower deviance is better).
```


Structured effects
−1.0
−0.5
0.0
0.5
1.0
JAGS
INLA
f(12)
f(11)
f(10)
f(9)
f(8)
f(7)
f(6)
f(5)
f(4)
f(3)
f(2)
f(1)


Posterior distribution for $\sigma = \tau^{-\frac{1}{2}}$
JAGS
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Density
0.0
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1.0
Standard deviation for the structured effect, σ

Example — Binary data with individual random effect

- R-INLA allows to make predictive inference based on the observed model
- Suppose for example that the $(n+1)-$th value is not (yet) observed for the response variable y
  - **NB**: for R-INLA, a missing value in the response means no likelihood contribution

Example — Binary data with individual random effect

- R-INLA allows to make predictive inference based on the observed model
- Suppose for example that the $(n+1)-$th value is not (yet) observed for the response variable y
  - **NB**: for R-INLA, a missing value in the response means no likelihood contribution

- We can code this in R, by augmenting the original dataset

```
y[n+1] <- NA
Ntrials[n+1] <- sample(c(80:100),size=1,replace=TRUE)
data2 <- data.frame(y=y,z=1:(n+1),Ntrials=Ntrials)

formula2 = y ~ f(z,model="iid",hyper=list(list(prior="flat")))
m2=inla(formula2,data=data2,
        family="binomial",
        Ntrials=Ntrials,
        control.predictor = list(compute = TRUE))

summary(m2)
```

Example — Binary data with individual random effect

```
Time used:
 Pre-processing    Running inla Post-processing           Total
         0.0883          0.0285          0.0236          0.1404
        (0.2258)        (0.0263)        (0.0744)        (0.3264)

 Fixed effects:
               mean      sd  0.025quant  0.5quant  0.975quant  kld
(Intercept) -0.0021   0.136      -0.272   -0.0021       0.268    0
           (-0.0021) (0.136)    (-0.272) (-0.0021)     (0.268)  (0)

Random effects:
Name   Model
 z   IID model

Model hyperparameters:
                mean     sd     0.025quant 0.5quant 0.975quant
Precision for z  7.130   4.087   2.168      6.186    17.599
                (7.130) (4.087) (2.168)    (6.168) (17.599)

Expected number of effective parameters(std dev): 9.494(0.7925)
Number of equivalent replicates : 1.264

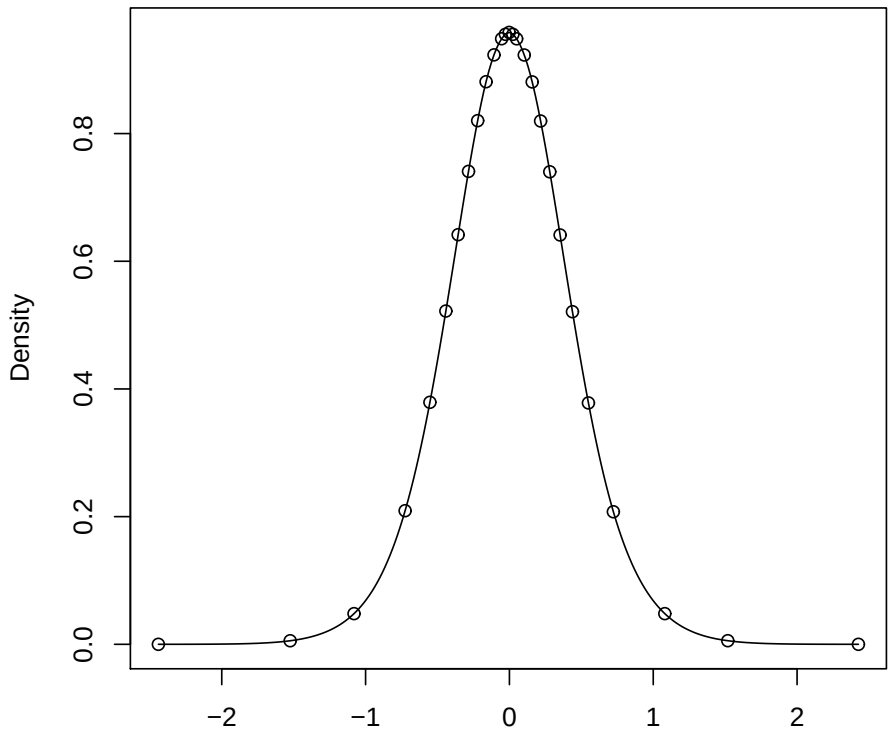
Marginal Likelihood:  -54.28
CPO and PIT are computed

Posterior marginals for linear predictor and fitted values computed
```

- The estimated value for the predictive distribution can be retrieved using the following code

```
pred <- m2$marginals.linear.predictor[[n+1]]
plot(pred,xlab="",ylab="Density")
lines(inla.smarginal(pred))
```

which can be used to generate, eg a graph of the predictive density



Specifying the model — options

- It is possible to specify link functions that are different from the default used by R-INLA
- This is done by specifying suitable values for the option control.family to the call to inla, eg

```
m = inla(formula, data=data, family="binomial", Ntrials=Ntrials,
         control.predictor=list(compute=TRUE),
         control.family = list(link = "probit"))
```

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- More details are available on the R-INLA website:
  - http://www.r-inla.org/models/likelihoods

Specifying the model — options

- R-INLA has a set of default priors for the different components of the LGM/GMRF
- For example, in a standard hierarchical formulation, R-INLA assumes
  - Unstructured ("fixed") effects: $\beta \sim \text{Normal}(0, 0.001)$
  - Structured ("random") effects: $f(z_i) \sim \text{Normal}(0, \tau)$
    $\log \tau \sim \text{logGamma}(1, 0.00005)$
- **NB**: It is possible to see the default settings using the function

```
inla.model.properties(<name>, <section>)
```

Specifying the model — options

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    $\log \tau \sim \text{logGamma}(1, 0.00005)$
- **NB**: It is possible to see the default settings using the function

```
inla.model.properties(<name>, <section>)
```

- However, there is a wealth of possible formulations that the user can specify, especially for the hyperpriors
- More details are available on the R-INLA website:
  - http://www.r-inla.org/models/likelihoods
  - http://www.r-inla.org/models/latent-models
  - http://www.r-inla.org/models/priors

Models for the observed data

Model	Name
Negative Binomial	nbinomial
Poisson	poisson
Binomial	binomial
Clustered Binomial	cbinomial
Gaussian	gaussian
Skew Normal	sn
Laplace	laplace
Student-t	T
Gaussian model for stochastic volatility	stochvol
Student-t model for stochastic volatility	stochvol.t
NIG model for stochastic volatility	stochvol.nig
Zero inflated Poisson	zeroinflated.poisson.x (x=0,1,2)
Zero inflated Binomial	zeroinflated.binomial.x (x=0,1)
Zero inflated negative Binomial	zeroinflated.nbinomial.x (x=0,1,2)
Zero inflated beta binomial (type 2)	zeroinflated.betabinomial.2
Generalised extreme value distribution (GEV)	gev
Beta	beta
Gamma	gamma
Beta-Binomial	betabinomial
Logistic distribution	logistic
Exponential (Survival models)	exponential
Weibull (Survival model)	weibull
LogLogistic (Survival model)	loglogistic
LogNormal (Survival model)	lognormal
Cox model (Survival model)	coxph

Models for the GMRF

Model	Name
Independent random variables	iid
Linear	linear
Random walk of order 1	rw1
Random walk of order 2	rw2
Continuous random walk of order 2	crw2
Model for seasonal variation	seasonal
Model for spatial effect	besag
Model for spatial effect	besagproper
Model for weighted spatial effects	besag2
Model for spatial effect + random effect	bym
Autoregressive model of order 1	ar1
Autoregressive model of order p	ar
The Ornstein-Uhlenbeck process	ou
User defined structure matrix, type 0	generic0
User defined structure matrix, type1	generic1
User defined structure matrix, type2	generic2
Model for correlated effects with Wishart prior (dimension 1, 2, 3, 4 and 5).	iid1d, iid2d, iid3d, iid4d, iid5d
(Quite) general latent model	z
Random walk of 2nd order on a lattice	rw2d
Gaussian field with Matern covariance function	matern2d
Classical measurement error model	mec
Berkson measurement error model	meb

Models for the hyper-parameters

Model	Name
Normal distribution	normal, gaussian
Log-gamma distribution	loggamma
Improper flat prior	flat
Truncated Normal distribution	logtnormal, logtgaussian
Improper flat prior on the log scale	logflat
Improper flat prior on the 1/ log scale	logiflat
Wishart prior	wishart
Beta for correlations	betacorrelation
Logit of a Beta	logitbeta
Define your own prior	expression:

Internal vs user scale

- Hyper-parameters (eg correlation coefficients ρ or precisions τ) are represented internally using a suitable transformation, eg

$$\psi_1 = \log(\tau)$$

or

$$\psi_2 = \log\left(\frac{1+\rho}{1-\rho}\right)$$

to improve symmetry and approximate Normality

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- Hyper-parameters (eg correlation coefficients ρ or precisions τ) are represented internally using a suitable transformation, eg

$$\psi_1 = \log(\tau)$$

or

$$\psi_2 = \log\left(\frac{1+\rho}{1-\rho}\right)$$

to improve symmetry and approximate Normality

- Initial values are given on the internal scale
- Priors are also defined on the internal scale
- So, when specifying custom values, care is needed!

Specifying the prior (1)

Consider the model

$$
\begin{aligned}
y_i \mid \theta_i, \sigma^2 &\sim \text{Normal}(\theta_i, \sigma^2) \\
\theta_i &= \alpha + \beta x_i \\
\alpha, \beta &\overset{iid}{\sim} \text{Normal}(0, 0.001) \\
\log \tau = -\log \sigma^2 &\sim \text{logGamma}(1, 0.01)
\end{aligned}
$$

```
n=100
a = 1; b = 1
x = rnorm(n)
eta = a + b*x
tau = 100
scale = exp(rnorm(n))
prec = scale*tau
y = rnorm(n, mean = eta, sd = 1/sqrt(prec))
data = list(y=y, x=x)
formula = y ~ 1 + x
result = inla(formula, family = "gaussian", data = data,
              control.family = list(hyper = list(
                 prec = list(prior = "loggamma",param = c(1,0.01),initial = 2))),
              scale=scale, keep=TRUE)
summary(result)
```

Specifying the prior (1)

```
Time used:
 Pre-processing    Running inla Post-processing           Total
         0.0776          0.0828          0.0189          0.1793

Fixed effects:
              mean     sd 0.025quant 0.5quant 0.975quant kld
(Intercept) 1.0013 0.0074     0.9868   1.0013     1.0158   0
x           0.9936 0.0075     0.9788   0.9936     1.0083   0

The model has no random effects

Model hyperparameters:
                                          mean    sd    0.025quant 0.5quant 0.975quant
Precision for the Gaussian observations 108.00 15.34 80.60       107.09   140.74

Expected number of effective parameters(std dev): 2.298(0.0335)
Number of equivalent replicates : 43.52

Marginal Likelihood:  83.86
CPO and PIT are computed
```

Specifying the prior (2)

Consider the model

$$
\begin{aligned}
y_i \mid \mu, \sigma^2 &\sim \text{Normal}(\mu, \sigma^2) \\
\mu &\sim \text{Normal}(0, 0.001) \\
\log \tau = -\log \sigma^2 &\sim \text{Normal}(0, 1)
\end{aligned}
$$

```
n = 10
y = rnorm(n)
formula = y ~ 1
result = inla(formula, data = data.frame(y),
              control.family = list(hyper = list(
                 prec = list(prior = "normal",param = c(0,1))))
)
summary(result)
```

Specifying the prior (2)

Consider the model

$$\begin{aligned} y_i \mid \mu, \sigma^2 &\sim \text{Normal}(\mu, \sigma^2) \\ \mu &\sim \text{Normal}(0, 0.001) \\ \log \tau = -\log \sigma^2 &\sim \text{Normal}(0, 1) \end{aligned}$$

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n = 10
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formula = y ~ 1
result = inla(formula, data = data.frame(y),
              control.family = list(hyper = list(
                 prec = list(prior = "normal",param = c(0,1))))
)
summary(result)
```

- **NB**: INLA thinks in terms of LGMs and GMRFs
- Thus, the common mean for all the observations is specified in terms of a regression!

Specifying the prior (2)

```
Time used:
 Pre-processing    Running inla Post-processing           Total
         0.0740          0.0214          0.0221          0.1175

Fixed effects:
               mean     sd 0.025quant 0.5quant 0.975quant kld
(Intercept) -0.3853 0.4077    -1.1939  -0.3853     0.4237   0

The model has no random effects

Model hyperparameters:
                                          mean    sd    0.025quant 0.5quant 0.975quant
Precision for the Gaussian observations 0.6512 0.268 0.2590     0.6089   1.2919

Expected number of effective parameters(std dev): 1.00(0.00)
Number of equivalent replicates : 9.999

Marginal Likelihood:  -17.30
CPO and PIT are computed
```

Specifying the prior (2)

Running the model in JAGS

```
model {
    for (i in 1:n) {
        y[i] ~ dnorm(mu,tau)
    }
    mu ~ dnorm(0,0.001)
    log.tau ~ dnorm(0,1)
    tau <- exp(log.tau)
}
```

produces similar results

```
Inference for Bugs model at "modelHyperPriorNormal.txt", fit using jags,
 2 chains, each with 1e+05 iterations (first 9500 discarded), n.thin = 181
 n.sims = 1000 iterations saved
         mu.vect sd.vect   2.5%  97.5%  Rhat n.eff
mu        -0.384   0.447 -1.293  0.555 1.000  1000
tau        0.642   0.258  0.233  1.240 1.006  1000
deviance  34.507   1.930 32.645 39.897 1.002   650

For each parameter, n.eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor (at convergence, Rhat=1).

DIC info (using the rule, pD = var(deviance)/2)
pD = 1.9 and DIC = 36.4
DIC is an estimate of expected predictive error (lower deviance is better).
```

Specifying the prior (2)

- We can also assume different priors for the unstructured ("fixed") effects, eg suppose we want to fit the model

$$\begin{aligned} y_i \mid \mu, \sigma^2 &\sim \text{Normal}(\mu, \sigma^2) \\ \mu &\sim \text{Normal}(10, 4) \\ \log \tau = -\log \sigma^2 &\sim \text{Normal}(0, 1) \end{aligned}$$

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- This can be done by using the option `control.fixed`, eg

```
result = inla(formula, data = data.frame(y),
              control.family = list(hyper = list(
                 prec = list(prior = "normal",param = c(0, 1))))
              control.fixed=list(
                 mean.intercept=10,prec.intercept=4)
)
```

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result = inla(formula, data = data.frame(y),
              control.family = list(hyper = list(
                 prec = list(prior = "normal",param = c(0, 1))))
              control.fixed=list(
                 mean.intercept=10,prec.intercept=4)
)
```

- **NB**: If the model contains fixed effects for some covariates, non-default priors can be included using the option

```
control.fixed=list(mean=list(value),prec=list(value))
```

Specifying the prior (2)

```
Time used:
 Pre-processing    Running inla Post-processing           Total
         0.0747          0.0311          0.0164          0.1222

Fixed effects:
              mean    sd 0.025quant 0.5quant 0.975quant kld
(Intercept) 9.5074 0.502     8.5249   9.5067    10.4935   0
           -0.3853 0.407    -1.1939  -0.3853     0.4237   0

The model has no random effects

Model hyperparameters:
                                          mean    sd    0.025quant 0.5quant 0.975quant
Precision for the Gaussian observations 0.0218 0.007 0.0105     0.0208   0.0391
                                        0.6512 0.268 0.2590     0.6089   1.2919

Expected number of effective parameters(std dev): 0.0521(0.0129)
                                                  1.0000(0.0000)
Number of equivalent replicates : 192.05
                                  9.9999

Marginal Likelihood:  153.84
                     -17.30
CPO and PIT are computed
```

Improving the estimation of the hyperparameters

- As mentioned earlier, for computational reasons, by default INLA uses a relatively rough grid to estimate the marginal posterior for the hyperparameters $p(\boldsymbol{\psi} \mid \boldsymbol{y})$
- This is generally good enough, but the procedure can be refined

Improving the estimation of the hyperparameters

- As mentioned earlier, for computational reasons, by default INLA uses a relatively rough grid to estimate the marginal posterior for the hyperparameters $p(\boldsymbol{\psi} \mid \boldsymbol{y})$
- This is generally good enough, but the procedure can be refined

- After the model has been estimated using the standard procedure, it is possible to increase precision in the estimation by re-fitting it using the command

```
inla.hyperpar(m, options)
```

- This modifies the estimation for the hyperparameters and (potentially, but not necessarily!) that for the parameters

A more complex model

- Consider the classic model for seizure counts in a RCT of anti-conversant therapy in epilepsy ("Epil" in the BUGS manual)
- The data are as follows

Patient	Visit 1	Visit 2	Visit 3	Visit 4	Trt	Base	Age
1	5	3	3	3	0	11	31
2	3	5	3	3	0	11	30
...	...	...	...	...	...	...	...
59	1	4	3	2	1	12	37

- We replicate the model presented in the BUGS manual, which uses slightly modified version of the covariates

A more complex model

We model

$$
\begin{aligned}
y_{jk} &\sim \text{Poisson}(\mu_{jk}) \\
\log(\mu_{jk}) &= \alpha_0 + \alpha_1 \log(B_j/4) + \alpha_2 Trt_j + \\
&\quad \alpha_3 Trt_j \times \log(B_j/4) + \alpha_4 \log(Age_j) + \\
&\quad \alpha_5 V4_k + u_j + v_{ik} \\
\alpha_0, \ldots \alpha_5 &\overset{iid}{\sim} \text{Normal}(0, \tau_\alpha), \qquad \tau_\alpha \text{ known} \\
u_j &\sim \text{Normal}(0, \tau_u), \qquad \tau_u \sim \text{Gamma}(a_u, b_u) \\
v_{jk} &\sim \text{Normal}(0, \tau_v), \qquad \tau_v \sim \text{Gamma}(a_v, b_v)
\end{aligned}
$$

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\end{aligned}
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$\boldsymbol{\alpha} = (\alpha_0, \ldots \alpha_5)$ indicates a set of "fixed" effects for the relevant (re-scaled) covariates

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u_j is an individual "random" effect

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\end{aligned}
$$

$\boldsymbol{\alpha} = (\alpha_0, \ldots \alpha_5)$ indicates a set of "fixed" effects for the relevant (re-scaled) covariates

u_j is an individual "random" effect

v_{jk} is a subject by visit "random" effect, which accounts for extra-Poisson variability

A more complex model

```
data(Epil)
head(Epil,n=3)
     y Trt Base Age V4 rand Ind
   1 5   0   11  31  0    1   1
   2 3   0   11  31  0    2   1
   3 3   0   11  31  0    3   1

formula <- y ~ log(Base/4) + Trt +
          I(Trt * log(Base/4)) + log(Age) + V4 +
          f(Ind, model = "iid") + f(rand, model="iid")

m <- inla(formula, family="poisson", data = Epil)
```

- **NB**: The variable `Ind` indicates the individual random effect u_j, while the variable `rand` is used to model the subject by visit random effect v_{jk}
- Interactions can be indicated in the R formula using the notation

```
I(var1 * var2)
```

- The model assumes that the two structured effects are independent. This can be relaxed and a joint model can be used instead

A more complex model

```
 Pre-processing    Running inla Post-processing           Total
         0.3672          0.2780           0.1276          0.7728

Fixed effects:
                       mean     sd 0.025quant 0.5quant 0.975quant    kld
(Intercept)         -1.3877 1.2107    -3.7621  -1.3913     1.0080 0.0055
log(Base/4)          0.8795 0.1346     0.6144   0.8795     1.1447 0.0127
Trt                 -0.9524 0.4092    -1.7605  -0.9513    -0.1498 0.0021
I(Trt * log(Base/4)) 0.3506 0.2081    -0.0586   0.3504     0.7611 0.0011
log(Age)             0.4830 0.3555    -0.2206   0.4843     1.1798 0.0007
V4                  -0.1032 0.0853    -0.2705  -0.1032     0.0646 0.0003

Random effects:
Name    Model
 Ind    IID model
rand    IID model

Model hyperparameters:
                  mean    sd     0.025quant 0.5quant 0.975quant
Precision for Ind   4.635  1.343  2.591       4.436     7.808
Precision for rand  8.566  2.115  5.206       8.298    13.458

Expected number of effective parameters(std dev): 118.97(8.586)
Number of equivalent replicates : 1.984

Marginal Likelihood:  -670.55
```

A more complex model

```
 Pre-processing    Running inla Post-processing           Total
         0.3672          0.2780           0.1276          0.7728
                                                (MCMC: approximately 30 mins)
Fixed effects:
                        mean     sd 0.025quant 0.5quant 0.975quant    kld
(Intercept)          -1.3877 1.2107    -3.7621  -1.3913     1.0080 0.0055
log(Base/4)           0.8795 0.1346     0.6144   0.8795     1.1447 0.0127
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  - Estimation time can be much lower than for standard MCMC
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- MCMC still provides a slightly more flexible approach
  - Virtually any model can be fit using `JAGS`/`BUGS`
  - The range of priors available is wider in an MCMC setting than in `INLA`
  - Documentation and examples is more extensive for standard MCMC models

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- Integrated Nested Laplace Approximation is a very effective tool to estimate LGMs
  - Estimation time can be much lower than for standard MCMC
  - Precision of estimation is usually higher than for standard MCMC
- MCMC still provides a slightly more flexible approach
  - Virtually any model can be fit using `JAGS`/`BUGS`
  - The range of priors available is wider in an MCMC setting than in `INLA`
  - Documentation and examples is more extensive for standard MCMC models
- Nevertheless, `INLA` proves to be a very flexible tool, which is able to fit a very wide range of models
  - Generalised linear (mixed) models
  - Log-Gaussian Cox processes
  - Survival analysis
  - Spline smoothing
  - Spatio-temporal models
- The `INLA` setup can be highly specialised (choice of data models, priors and hyperpriors) although this is a bit less intuitive than (most) MCMC models

Thank you!