The quantile probability model

A novel approach to Bayesian variable selection for clinical prediction

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Clinical prediction models

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- ⇒ to identify low and high risk patients for a specific disease / pathogen / event /...
- ⇒ to better target treatment and prevention
- ⇒ to improve medical practice

Objective Bayesian variable selection for GLMs

Select among p variables $\rightarrow$ 2^p candidate models $\mathcal{M}_j$

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Posterior model probability:

$$\Pr(\mathcal{M}_j \mid \text{data}) = \frac{\text{TBF}_{j,0}\Pr(\mathcal{M}_j)}{\sum_{k \in \mathcal{J}} \text{TBF}_{k,0}\Pr(\mathcal{M}_k)}$$

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Generalized g-prior [1,2] leads to

$$\text{TBF}_{j,0} = \frac{\text{p}(z_j \mid \mathcal{M}_j)}{\text{p}(z_j \mid \mathcal{M}_0)} = (g+1)^{-d_j/2} \exp\left(\frac{g}{g+1}\frac{z_j}{2}\right)$$

z_j is the deviance statistic of model $\mathcal{M}_j$ with d_j degrees of freedom

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Posterior inclusion probability of x_k, with $k \in \{1, \ldots, p\}$:

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BUT MPM also highly sentitive to prior choices

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Prior choices

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$\Rightarrow$ fixed prior inclusion probability at $q = 1/2$

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$\Rightarrow$ beta distibution on the prior inclusion probability $q \sim \mathrm{Be}(a, b)$

$$\rightarrow \Pr(\mathcal{M}_j \mid a, b) \quad = \quad \frac{\mathrm{B}(a + p_j, b + p - p_j)}{\mathrm{B}(a, b)}$$

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Definition of g

$\Rightarrow$ Local empirical Bayes: $\hat{g} = \max\{z_j/d_j - 1, 0\}$

$\Rightarrow$ hyper-g/n: $\frac{g/n}{g/n+1} \sim \mathrm{U}(0, 1)$

$\Rightarrow$ Zellner-Siow adapted

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Case study

Application on **GUSTO-I** data using logistic regression

$\Rightarrow$ Randomized study for comparison of 4 different treatments in over 40‘000 acute myocardial infarction patients

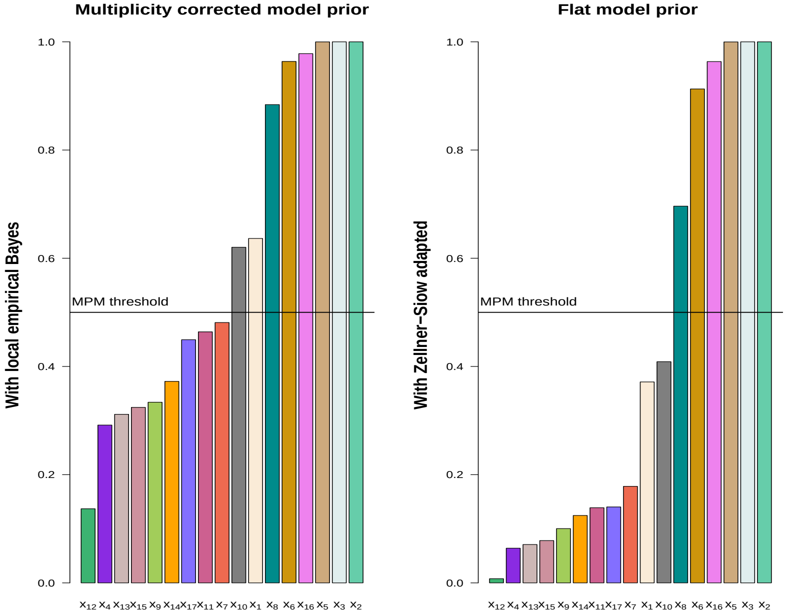
$\Rightarrow$ We use a publicly available subgroup from the Western region of the USA ($n = 2188$)

$\Rightarrow$ prediction of **30-days survival**

We focus on **17 covariates**

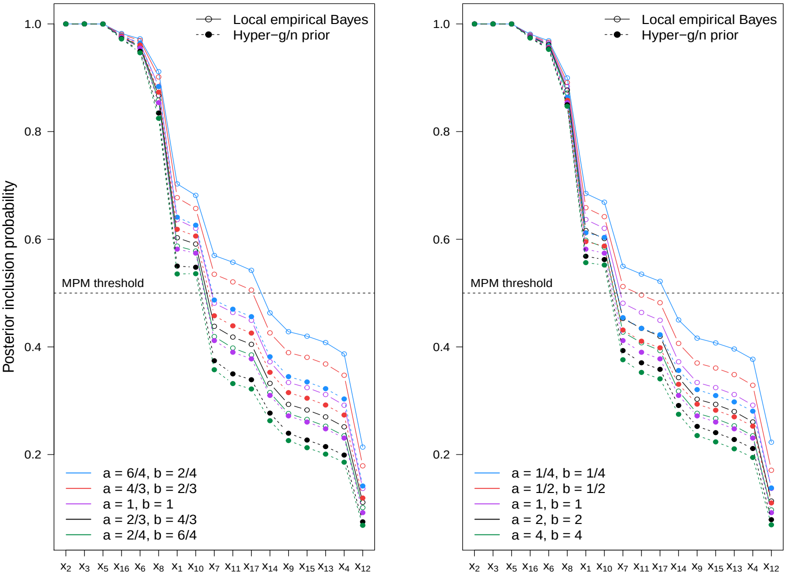
Sensitivity of the MPM

with respect to the definition of *g* and the model prior



Sensitivity of the MPM

with respect to the hyper-parameters *a* and *b*



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- Compute predictive model selection criterion:

  DIC, WAIC or LOO-IC
- Select the model with lowest information criterion $\rightarrow$ QPM

Predictive model selection information criteria (IC)

The **deviance information criterion** (DIC):

$$-2 \log f(\mathbf{y} \mid \hat{\theta}) \quad + \quad 2p_{\text{DIC}}$$

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→ IC does not depend on model prior !

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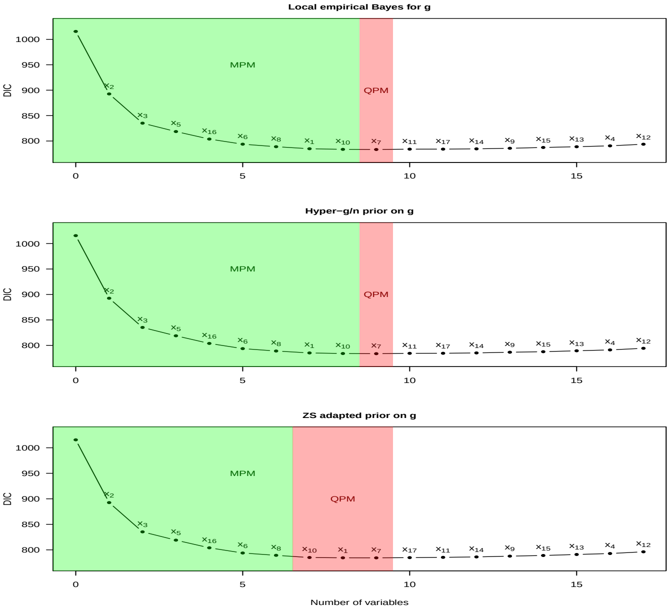
QPM model $\mathcal{M}_9$ independent from the definition of g



Monte Carlo error of DIC

Sampling from the posterior distribution $\rightarrow$ **Monte Carlo** (MC) **error**

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Brute force approach:

Replicate the DIC calculation N times and compute sample standard deviation of the N DIC estimates

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⇒ very time-consuming

New efficient approach for Monte Carlo error of DIC

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Suppose DIC has been computed using MC sample of size S

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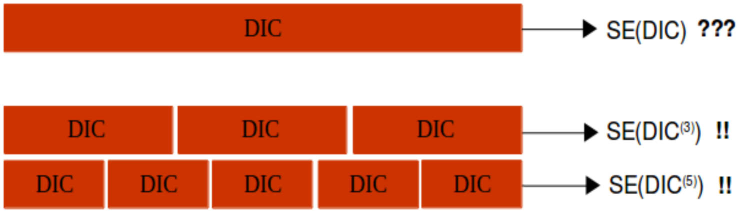

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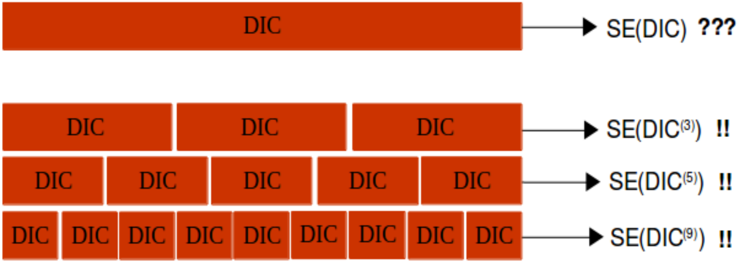

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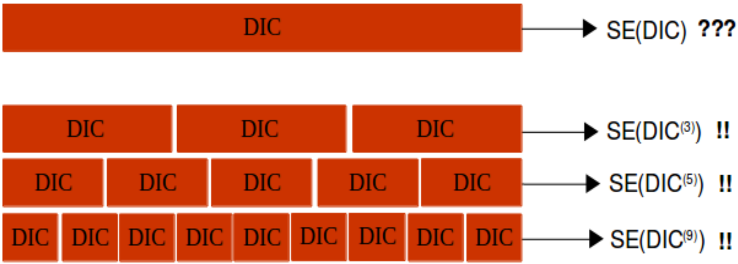

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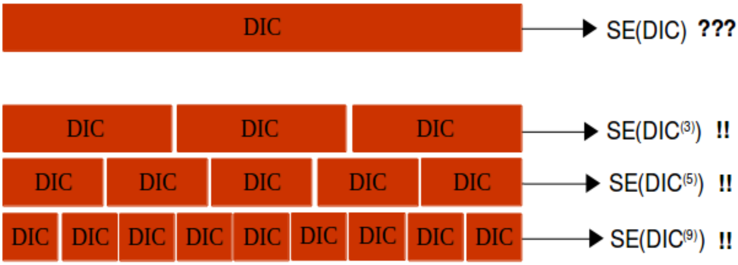

with the square-root law we obtain $\mathrm{SE}(\mathrm{DIC}^{(k)}) \propto \sqrt{\frac{1}{S/k}} \propto \sqrt{k}$

$\Rightarrow \mathrm{SE}(\mathrm{DIC}^{(k)}) = c \cdot \sqrt{k} \quad \Rightarrow \quad \mathrm{SE}(\mathrm{DIC}^{(1)}) = c$

estimate c with weighted linear regression of $\mathrm{SE}(\mathrm{DIC}^{(k)})$ on $\sqrt{k}$ with weights k

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Application to case study

Case study: $S = 50,000$ and $k \in \{80, 100, 125, 160, 200\}$

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The **Monte Carlo standard error** of $\Delta\text{DIC}_j = \text{DIC}_{j-1} - \text{DIC}_j$:

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	ΔDIC_j	(SE)	ΔDIC_j	(SE)	ΔDIC_j	(SE)
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$\mathcal{M}_8$ vs. $\mathcal{M}_9$	0.217	(0.031)	0.166	(0.034)	0.057	(0.033)
$\mathcal{M}_9$ vs. $\mathcal{M}_{10}$	-0.295	(0.031)	-0.342	(0.034)	-0.170	(0.032)
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Conclusion: The DIC of the QPM $\mathcal{M}_9$ is the unique minimum !

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- Work revised for *Journal of Computational Statistics and Data Analysis*

Thank you

Questions? Comments?