Bayesian Forecasting of Infectious Diseases with SIRS Models

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Mass Balance and SIRS Dynamics

- 'S' $\equiv$ Susceptible; 'I' $\equiv$ Infectious; 'R' $\equiv$ Recovered
- Mass balance: The classic SIR model assumes that there are no births and deaths from causes other than the disease itself. Thus, the numbers who are susceptible, infectious, and recovered satisfy,

$$S(t) + I(t) + R(t) = N$$

  where N is the size of the population. From the equation above,

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

- Temporal-only SIRS dynamics, where 'R' returns to 'S':

Classic SIRS (CSIRS) Model

- The SIRS (susceptible-infectious-recovered-susceptible) model is a type of compartment epidemic model that has been used widely to study the dynamics of infectious diseases in large populations (e.g., Anderson and May, 1991, OU Press).
- The Classic SIRS (CSIRS) Ordinary Differential Equations (ODEs) are nonlinear (where $\phi = 0$ gives the traditional SIR model):

$$\frac{dS}{dt} = -\beta SI + \phi R,$$

$$\frac{dI}{dt} = \beta SI - \gamma I,$$

$$\frac{dR}{dt} = \gamma I - \phi R,$$

- In the ODEs above, β, γ, and ϕ denote the transmission rate, the rate of recovery, and the rate of loss of immunity, respectively, in units of per day (d^{-1}).

HSIRS Model

Hierarchical SIRS (HSIRS) has a measurement model (Poisson) and a latent process model that satisfies the ODEs in the previous slide.

Notation:

- $\mathbf{Z}(t) \equiv (Z_S(t), Z_I(t), Z_R(t))'$: observed counts S, I, R at time t (noisy data).
- $\boldsymbol{\lambda}(t) \equiv (\lambda_S(t), \lambda_I(t), \lambda_R(t))'$: latent (mean) counts S, I, R at time t; $\lambda_N \equiv \lambda_S(t) + \lambda_I(t) + \lambda_R(t)$, a constant over time; $\lambda_S(t) \equiv \lambda_N P_S(t)$, $\lambda_I(t) \equiv \lambda_N P_I(t)$, $\lambda_R(t) \equiv \lambda_N P_R(t)$.
- $\mathbf{P}(t) \equiv (P_S(t), P_I(t), P_R(t))'$: latent rates of S, I, R at time t, and $P_R(t) = 1 - P_S(t) - P_I(t)$.
- $\mathbf{W}(t) \equiv (W_S(t), W_I(t))'$: hidden log odds ratios of the true rates of S/R, I/R at time t.
- Parameters $\boldsymbol{\theta}$ include β, γ, ϕ, and log-odds-ratio variances.

The Latent Dynamical Process in HSIRS

From the SIRS ODEs, the deterministic difference equations on the latent mean counts $\lambda_S(t)$, $\lambda_I(t)$, and $\lambda_R(t)$, for discrete time $t = 1, 2, \dots$ in units of Δ days, are:

$$\lambda_S(t+1) = \lambda_S(t) - \beta\Delta\lambda_S(t)\lambda_I(t) + \phi\Delta\lambda_R(t),$$

$$\lambda_I(t+1) = \lambda_I(t) + \beta\Delta\lambda_S(t)\lambda_I(t) - \gamma\Delta\lambda_I(t),$$

$$\lambda_R(t+1) = \lambda_R(t) + \gamma\Delta\lambda_I(t) - \phi\Delta\lambda_R(t).$$

Since a general latent rate satisfies $P(t) \equiv \lambda(t)/\lambda_N$,

$$P_S(t+1) = P_S(t) - \beta\Delta\lambda_N P_S(t)P_I(t) + \phi\Delta P_R(t),$$

$$P_I(t+1) = P_I(t) + \beta\Delta\lambda_N P_S(t)P_I(t) - \gamma\Delta P_I(t),$$

$$P_R(t+1) = P_R(t) + \gamma\Delta P_I(t) - \phi\Delta P_R(t).$$

Fully Bayesian SIRS (we call it ASIRS)

Data Model (variability in observed S, I, and R)
The measurement is assumed Poisson distributed. For a generic $\{Z(t)\}$ and $\{P(t)\}$:

$$Z(t)|P(t) \sim \text{ind. Poisson}(\lambda_N P(t))$$

Process Model

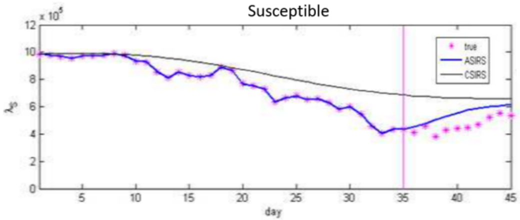
- $W_S(t) \equiv \log\left(\frac{P_S(t)}{P_R(t)}\right), W_I(t) \equiv \log\left(\frac{P_I(t)}{P_R(t)}\right)$ (transform to **W**-scale)
- $P_S(t) + P_I(t) + P_R(t) = 1$ (mass balance)
- $\mathbf{W}(t+1) = \boldsymbol{\mu}^W(t) + \boldsymbol{\xi}(t+1)$ (dynamics on **W**-scale)
- $\boldsymbol{\xi}(t) \sim \text{MVN}(\mathbf{0}, \text{diag}(\sigma^2_{\xi_S}, \sigma^2_{\xi_I}))$ (small-scale variation)

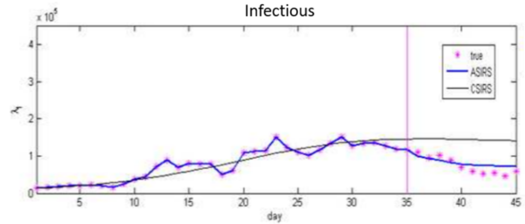
Parameter Model (prior information on parameters)

$$[\beta, \gamma, \phi, \sigma^2_{\xi_S}, \sigma^2_{\xi_I})]$$

Simulated Data: CSIRS (Classic) v. ASIRS (Bayesian)

Classic and Bayesian models fitted to days 1-35: smooth and forecast days 1-45





Conclusion

- Bayesian SIRS models account for uncertainty in the measurements (Data Model) and uncertainty in the parameters (Parameter Model, or prior). Bayesian SIRS models vastly outperform Classic SIRS models (see the example above and the simulation experiment in the paper).
- All details and more are available in:
  Zhuang, L. and Cressie, N. "Bayesian hierarchical statistical SIRS models." *Stat. Methods Appl.*, **23**, 601-646 (2014).
- Further, a multi-species SIR model that is dynamical and Bayesian was presented in:
  Zhuang, L., Cressie, N., Pomeroy, L., and Janies, D. "Multi-species SIR models from a dynamical Bayesian perspective." *Theor. Ecol.*, **6**, 457-473 (2013).
- These results are "temporal" only. Now go "spatio-temporal"!